1. Determine if each of the following are valid autocorrelation matrices of WSS processes. (Correlation Matrix)

\[ R_a = \begin{bmatrix} 4 & 1 & 1 \\ -1 & 4 & 1 \\ -1 & -1 & 4 \end{bmatrix}, \quad R_b = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \quad R_c = \begin{bmatrix} 2j & 0 & j \\ 0 & 2j & 0 \\ -j & 0 & 2j \end{bmatrix}, \quad R_d = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}. \]

2. Consider the random process \( y(n) = x(n) + v(n) \), where \( x(n) = Ae^{j(\omega n + \phi)} \) and \( v(n) \) is zero mean white Gaussian noise with a variance \( \sigma_v^2 \). We also assume the noise and the complex sinusoid are independent. Under the following conditions, determine if \( y(n) \) is WSS. Justify your answers. (WSS Process)

(a) \( \omega \) and \( A \) are constants, and \( \phi \) is a uniformly distributed over the interval \([0, 2\pi] \).

(b) \( \omega \) and \( \phi \) are constants, and \( A \) is a Gaussian random variable \( \sim N(0, \sigma_A^2) \).

(c) \( \phi \) and \( A \) are constants, and \( \omega \) is a uniformly distributed over the interval \([\omega_0 - \Delta, \omega_0 + \Delta] \) for some fixed \( \Delta \).

3. [Rec.II P2(a) revisited] Determine the PSD of the WSS process \( y(n) = Ae^{j(\omega n + \phi)} + v(n) \), where \( v(n) \) is zero mean white Gaussian noise with a variance \( \sigma_v^2 \), and \( \phi \) is uniformly distributed over the interval \([0, 2\pi] \). (Power Spectral Density)

4. Assume \( v(n) \) is a white Gaussian random process with zero mean and variance 1. The two filters in Fig. RII.4 are \( G(z) = \frac{1}{1-0.4z^{-1}} \) and \( H(z) = \frac{2}{1-0.3z^{-1}} \). (Auto-Regressive Process)

\[ \begin{align*}
&G(z) \quad \vdots \quad H(z) \\
v(n) &\quad \longrightarrow & u(n)
\end{align*} \]

Figure RII.4:

(a) Is \( u(n) \) an AR process? If so, find the parameters.

(b) Find the autocorrelation coefficients \( r_u(0), r_u(1), \) and \( r_u(2) \) of the process \( u(n) \).

5. Let a real-valued AR(2) process be described by

\[ u(n) = x(n) + a_1x(n-1) + a_2x(n-2) \]

where \( u(n) \) is a white noise of zero-mean and variance \( \sigma^2 \), and \( u(n) \) and past values \( x(n-1), x(n-2) \) are uncorrelated. (Yule-Walker Equation)
(a) Determine and solve the Yule-Walker Equations for the AR process.
(b) Find the variance of the process \( x(n) \).

6.   [Problem 5.3 continued] Assume \( v(n) \) and \( w(n) \) are white Gaussian random processes with zero mean and variance 1. The two filters in Fig. RII.6 are \( G(z) = \frac{1}{1 - 0.42z^{-1}} \) and \( H(z) = \frac{2}{1 - 0.52z^{-1}} \). (Wiener Filter)

\[
\begin{array}{c}
\text{v(n)} \\
\text{G(z)} \\
\text{H(z)} \\
\text{u(n)} \\
\text{x(n)} \\
\text{Winner Filter} \\
\text{y(n)}
\end{array}
\]

(a) Design a 1-order Wiener filter such that the desired output is \( u(n) \). What is the MSE?
(b) Design a 2-order Wiener filter. What is the MSE?

7.   The autocorrelation sequence of a given zero-mean real-valued random process \( u(n) \) is \( r(0) = 1.25, r(1) = r(-1) = 0.5, \) and \( r(k) = 0 \) for any \( |k| \geq 2 \). (Wiener Filter)

(a) What model fits this process best: AR or MA? Find the corresponding parameters.
(b) Design the Wiener filter when using \( u(n) \) to predict \( u(n + 1) \). Can we do better (in terms of MSE) if we use both \( u(n) \) and \( u(n - 1) \) as the input to the Wiener filter? What if using \( u(n) \) and \( u(n - 2) \)?

8.   Consider the MIMO (multi-input multi-output) wireless communications system shown in Fig. RII.8. There are two antennas at the transmitter and three antennas at the receiver. Assume the channel gain from the \( i \)-th transmit antenna to the \( j \)-th receive antenna is \( h_{ji} \). Take a snapshot at time slot \( n \), the received signal is \( y_j(n) = h_{j1}x_1(n) + h_{j2}x_2(n) + v_j(n) \) where \( v_j(n) \) are white Gaussian noise (zero mean, variance \( N_0 \)) independent of signals. We further assume \( x_1(n) \) and \( x_2(n) \) are uncorrelated, and their power are \( P_1 \) and \( P_2 \), respectively. Use \( y_1(n), y_2(n) \) and \( y_3(n) \) as input, find the optimal Wiener filter to estimate \( x_1(n) \) and \( x_2(n) \). (Wiener Filter)

\[
\begin{array}{c}
x_1(n) \\
x_2(n) \\
x_1(n) \\
x_2(n) \\
x_1(n) \\
x_2(n)
\end{array}
\]

\[
\begin{array}{c}
\text{Wiener Filter} \\
\text{\hat{x}_1(n)} \\
\text{\hat{x}_2(n)} \\
\text{\hat{x}_1(n)} \\
\text{\hat{x}_2(n)}
\end{array}
\]

Figure RII.8:
9. Given a real-valued AR(3) model with parameters $\Gamma_1 = -4/5$, $\Gamma_2 = 1/9$, $\Gamma_3 = 1/8$, and $r(0) = 1$, find $r(1), r(2),$ and $r(3)$. (Levinson-Durbin Recursion)

10. Consider the MA(1) process $x(n) = v(n) + bv(n - 1)$ with $v(n)$ being a zero-mean white sequence with variance 1. If we use $\Gamma_k$ to represent this system, prove that (Levinson-Durbin Recursion)

$$\Gamma_{m+1} = \frac{\Gamma_m^2}{\Gamma_{m-1}(1 - |\Gamma_m|^2)}.$$

11. Given a $p$-order AR random process $\{x(n)\}$, it can be equivalently represented by any of the three following sets of values: (Levinson-Durbin Recursion)

- $\{r(0), r(1), \ldots, r(p)\}$
- $\{a_1, a_2, \ldots, a_p\}$ and $r(0)$
- $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_p\}$ and $r(0)$

(a) If a new random process is defined as $x'(n) = cx(n)$ where $c$ is a real-valued constant, what will be the new autocorrelation sequence $r'(k)$ in terms of $r(k)$ (for $k = 1, 2, \ldots, p$)? How about $a'_k$ and $\Gamma'_k$?

(b) Let a new random process be defined as $x'(n) = (-1)^n x(n)$. Prove that $r'(k) = (-1)^k r(k)$, $a'_k = (-1)^k a_k$ and $\Gamma'_k = (-1)^k \Gamma_k$. (Hint: use induction when proving $\Gamma_k$, since $\Gamma_k$ is calculated recursively.)

12. Given a lattice predictor that simultaneously generates both forward and backward prediction errors $f_m(n)$ and $b_m(n)$ ($m = 1, 2, \ldots, M$). (Lattice Structure)

(a) Find $E(f_m(n)b^*_i(n))$ for both conditions when $i \leq m$ and $i > m$.

(b) Find $E(f_{m+n}(n + i)f^*_i(n))$ for both conditions when $i = m$ and $i < m$.

(c) Design a joint process estimation scheme using the forward prediction errors.

(d) If for some reason we can only obtain part of forward prediction error (from order 0 to order $k$) and part of backward prediction error (from order $k + 1$ to order $M$), i.e., we have $\{f_0(n), f_1(n), \ldots, f_k(n), b_{k+1}(n), b_{k+2}(n), \ldots, b_M(n)\}$. Describe how to use such mixed forward and backward prediction errors to perform joint process estimation.

(Hint: the results from (a) and (b) will be useful for questions (c) and (d).)

13. Consider the backward prediction error sequence $b_0(n), b_1(n), \ldots, b_M(n)$ for the observed sequence $\{u(n)\}$. (Properties of FLP and BLP Errors)
(a) Define $b(n) = [b_0(n), b_1(n), \ldots, b_M(n)]^T$, and $u(n) = [u(n), u(n-1), \ldots, u(n-M)]^T$, find $L$ in terms of the coefficients of the backward prediction-error filter where $b(n) = Lu(n)$.

(b) Let the correlation matrix for $b(n)$ be $D$, and that for $u(n)$ be $R$. Is $D$ diagonal? What is relation between $R$ and $D$? Show that a lower triangular matrix $A$ exists such that $R^{-1} = A^H A$.

(c) Now we are to perform joint estimation of a desired sequence $\{d(n)\}$ by using either $\{b_k(n)\}$ or $\{u(n)\}$, and their corresponding optimal weight vectors are $k$ and $w$, respectively. What is relation between $k$ and $w$?