Problem 1 A first order autoregressive (AR) process \( \{u(n)\} \) that is real-valued satisfies the real-valued difference equation

\[
    u(n) + a_1 u(n - 1) = v(n)
\]

where \( a_1 \) is a constant and \( \{v(n)\} \) is a white-noise process of variance \( \sigma_v^2 \). Such a process is also referred to as a first-order Markov process.

(a) Suppose in practical implementation, the generation of the process \( \{u(n)\} \) starts at \( n=1 \) with initialization \( u(0)=0 \). Determine the mean of the actual \( \{u(n)\} \) process that we have obtained. Under what conditions \( E[u(n)] \) converges to a constant and what the constant is?

(b) Now consider the case when \( \{v(n)\} \) has zero mean. Determine the variance of the actual \( \{u(n)\} \) process that we have obtained. Under what conditions \( Var[u(n)] \) converges to a constant and what the constant is?

(c) For the conditions specified in part (b), find the autocorrelation function of the AR process \( \{u(n)\} \). Sketch this autocorrelation function when \( n \gg k \), for the two cases \( 0 < a_1 < 1 \) and \( -1 < a_1 < 0 \).

Problem 2 Consider an autoregressive process \( \{u(n)\} \) of order 2, described by the difference equation

\[
    u(n) = u(n - 1) - 0.5u(n - 2) + v(n)
\]

where \( \{v(n)\} \) is a white-noise process of zero mean and variance 0.5.
(a) Write the Yule-Walker equations for the process.

(b) Solve these two equations for the autocorrelation function values $r(1)$ and $r(2)$.

(c) Find the variance of $\{u(n)\}$.

**Problem 3** Consider an MA process $\{x(n)\}$ of order 2 described by the difference equation

$$x(n) = v(n) + 0.75v(n-1) + 0.25v(n-2)$$

where $\{v(n)\}$ is a zero mean white noise process of unit variance. The requirement is to approximate the process by an AR process $\{u(n)\}$ of order $M$. Do this approximation for the orders $M = 2$ and $M = 5$, respectively.

**Problem 4** A discrete-time stochastic process $\{x(n)\}$ that is real-valued consists of an AR process $\{u(n)\}$ and additive white noise process $\{v_2(n)\}$. The AR component is described by the difference equation

$$u(n) + \sum_{k=1}^{M} a_k u(n-k) = v_1(n)$$

where $\{a_k\}$ are the set of AR parameters and $\{v_1(n)\}$ is a white noise process that is independent of $\{v_2(n)\}$. Show that $\{x(n)\}$ is an ARMA process described by

$$x(n) = -\sum_{k=1}^{M} a_k x(n-k) + \sum_{k=1}^{M} b_k e(n-k) + e(n)$$

where $\{e(n)\}$ is a white noise process. How are the MA parameters $\{b_k\}$ defined? How is the variance of $e(n)$ defined?