Solution to 1 Since the system is alias-free,

\[
H(z)f(z) = \begin{pmatrix} M \, T(z) \\ 0 \\ . \\ . \\ 0 \end{pmatrix}
\]

In other words,

\[
H_0(z)F_0(z) + \cdots + H_{M-1}(z)F_{M-1}(z) = M \, T(z)
\]

\[
H_0(zW^i)F_0(z) + \cdots + H_{M-1}(zW^i)F_{M-1}(z) = 0, \quad i = 1, \cdots, M - 1
\]

If replace \( z \) with \( zW^{M-i} \), it’ll continue to hold:

\[
H_0(z)F_0(zW^{M-i}) + \cdots + H_{M-1}(z)F_{M-1}(zW^{M-i}) = 0, \quad i = 1, \cdots, M - 1
\]

Thus,

\[
\begin{pmatrix} F_0(z) & \cdots & F_{M-1}(z) \\ . & \ddots & . \\ . & . & . \\ F_0(zW^{M-1}) & \cdots & F_{M-1}(zW^{M-1}) \end{pmatrix} \begin{pmatrix} H_0(z) \\ \vdots \\ \vdots \\ H_{M-1}(z) \end{pmatrix} = \begin{pmatrix} M \, T(z) \\ 0 \\ \vdots \\ 0 \end{pmatrix}
\]

We can swap \( H_k(z) \)'s and \( F_k(z) \)'s and the system is still alias-free with the same \( T(z) \).

Solution to 2 a)

\[
H_k(z) = \sum_{i=0}^{M-1} (E_i(z^Mz^{-i})W^{-ki})
\]

\[
= \sum_{i=0}^{M-1} E_i((zW^k)^M(zW^k)^{-i})
\]

\[
= H_0(zW^k)
\]
where
\[ H_0(z) = \sum_{i=0}^{M-1} E_i(z^M) z^{-i} \]
\[ F_k(z) = \sum_{i=0}^{M-1} (R_i(z^M) z^{-(M-1-i)}) W^{ki} \]
\[ = W^{-k} \sum_{i=0}^{M-1} R_i((zW^k)^M)(zW^k)^{-(M-1-i)} \]
\[ = W^{-k} F_0(zW^k) \]

where we have used
\[ W^{ki} = W^{-k}(W^k)^{-(M-1-i)} \]
in going from line 1 to 2, and
\[ F_0(z) = \sum_{i=0}^{M-1} R_i(z^M) z^{-(M-1-i)} \]

b) The distortion function is
\[ T(z) = z^{-(M-1)} \prod_{i=0}^{M-1} E_i(z^M) \]

c) From part (a), the AC matrix is
\[
\begin{pmatrix}
H_0(z) & H_0(zW) & \cdots & H_0(zW^{M-1}) \\
H_0(zW) & H_0(zW^2) & \cdots & H_0(z) \\
\vdots & \ddots & \ddots & \vdots \\
H_0(zW^{M-1}) & H_0(z) & \cdots & H_0(zW^{M-2}) \\
\end{pmatrix}
\]

which is obviously a left circulant matrix.

d) Note that
\[ E(z^M) = W^* \text{diag}(E_i(z^M)) \]
Further note the relation between AC matrix and polyphase matrix given in the hint (see the proof in Vaidyanathan’s textbook pp234)

\[ H(z) = W^H D(z) E^T (z^M). \]

We have

\[ \det |H(z)| = cz^{-L} \prod_{i=0}^{M-1} E_i(z^M) \]

which is of the form \( cz^{-K} T(z) \).

e) Since \( E_L(\pi) = 0 \), \( E_L(w_i M) = 0 \) for \( w_i = \frac{\pi}{M} (2n + 1) \). Therefore, \( T(w) = 0 \) for \( w = \frac{\pi}{M}, \frac{3\pi}{M}, \ldots, \) etc. Let

\[ N + 1 = Mq + m, \quad 0 \leq m < M \]

There are 4 different possibilities depending upon \((q,m)\):

1. \((q,m) = (\text{odd, odd})\), \( T(w) = 0 \).
2. \((q,m) = (\text{odd, even})\), This problem is avoided.
3. \((q,m) = (\text{even, odd})\), This problem is avoided if \( M \) is odd.
4. \((q,m) = (\text{even, even})\), This problem is avoided if \( M \) is even.

Please check the symmetry and the order of each polyphase components for the four cases mentioned above. The main goal is to avoid the situation that a tap and its image are in the same polyphase component. In such situation, the polyphase component would be symmetric and the transfer function would become zero.

**Solution to 3** Note that

\[ T(z) = z^{-2} [P_0(z^3) + z^{-1} P_1(z^3) + z^{-2} P_2(z^3)] \]
where the expression inside the bracket is the polyphase decomposition of 
$T(z)$ and $z^{-2}$ is the delay ($=M-1$).

Now, by using

$$
\frac{1}{1-az^{-1}} = \frac{1 + a z^{-1} + a^2 z^{-2}}{(1 - az^{-1})(1 + az^{-1} + a^2 z^{-2})} = \frac{1 + a z^{-1} + a^2 z^{-2}}{1 - a^3 z^{-3}}
$$

the followings can be derived:

$$
P_0(z) = \frac{1}{1 - a^3 z^{-1}}, \quad P_1(z) = \frac{a}{1 - a^3 z^{-1}}, \quad P_2(z) = \frac{a^2}{1 - a^3 z^{-1}}
$$

Thus,

$$
\begin{align*}
P(z) &= \frac{1}{1 - a^3 z^{-1}} \left( \begin{array}{ccc}
1 & a & a^2 \\
1 & a & a \\
a^2 z^{-1} & 1 & a \\
z^{-1} & a^2 z^{-1} & 1 
\end{array} \right)
\end{align*}
$$

Solution to 4 a)

$$
\hat{X}(z) = \frac{1}{L} \sum_{l=0}^{L-1} [X(z W^l) \sum_{k=0}^{M-1} H_k(z W^l) F_k(z)]
$$

b) Consider Figure 1. Make sure that

$$
\frac{2\pi}{3} - \epsilon \geq \frac{2\pi}{4} + \epsilon
$$

So, $\epsilon \leq \frac{\pi}{12}$. To eliminate aliasing, we can pick $F_k(z)$ to look like those of $H_k(z)$. The transform function is

$$
T(z) = \frac{1}{3} \sum_{k=0}^{3} H_k(z) F_k(z)
$$
Figure 1: