Problem 5.24 :

(a) Since \( m_2(t) = -m_3(t) \) the dimensionality of the signal space is two.

(b) As a basis of the signal space we consider the functions:

\[
\begin{align*}
    f_1(t) &= \begin{cases} 
    \frac{1}{\sqrt{T}} & 0 \leq t \leq T \\
    0 & \text{otherwise}
    \end{cases} \\
    f_2(t) &= \begin{cases} 
    \frac{1}{\sqrt{T}} & 0 \leq t \leq \frac{T}{2} \\
    -\frac{1}{\sqrt{T}} & \frac{T}{2} < t \leq T \\
    0 & \text{otherwise}
    \end{cases}
\end{align*}
\]

The vector representation of the signals is:
\[
\begin{align*}
    m_1 &= [\sqrt{T}, \ 0] \\
    m_2 &= [0, \ \sqrt{T}] \\
    m_3 &= [0, -\sqrt{T}]
\end{align*}
\]

(c) The signal constellation is depicted in the next figure:

\[
\begin{align*}
    (0, \sqrt{T}) & \quad \bullet \\
    (\sqrt{T}, 0) & \quad \bullet \\
    (0, -\sqrt{T}) & \quad \bullet
\end{align*}
\]

(d) The three possible outputs of the matched filters, corresponding to the three possible transmitted signals are \((r_1, r_2) = (\sqrt{T} + n_1, n_2), (n_1, \sqrt{T} + n_2)\) and \((n_1, -\sqrt{T} + n_2)\), where \(n_1, n_2\) are zero-mean Gaussian random variables with variance \(\frac{N_0}{2}\). If all the signals are equiprobable the optimum decision rule selects the signal that maximizes the metric (see 5-1-44):

\[
C(r, m_i) = 2r \cdot m_i - |m_i|^2
\]

or since \(|m_i|^2\) is the same for all \(i\),

\[
C'(r, m_i) = r \cdot m_i
\]

Thus the optimal decision region \(R_1\) for \(m_1\) is the set of points \((r_1, r_2)\), such that \((r_1, r_2) \cdot m_1 > (r_1, r_2) \cdot m_2\) and \((r_1, r_2) \cdot m_1 > (r_1, r_2) \cdot m_3\). Since \((r_1, r_2) \cdot m_1 = \sqrt{T}r_1, (r_1, r_2) \cdot m_2 = \sqrt{T}r_2\) and \((r_1, r_2) \cdot m_3 = -\sqrt{T}r_2\), the previous conditions are written as

\[
    r_1 > r_2 \quad \text{and} \quad r_1 > -r_2
\]
Similarly we find that $R_2$ is the set of points $(r_1, r_2)$ that satisfy $r_2 > 0$, $r_2 > r_1$ and $R_3$ is the region such that $r_2 < 0$ and $r_2 < -r_1$. The regions $R_1$, $R_2$ and $R_3$ are shown in the next figure.

(e) If the signals are equiprobable then:

$$P(e|m_1) = P(|r - m_1|^2 > |r - m_2|^2|m_1) + P(|r - m_1|^2 > |r - m_3|^2|m_1)$$

When $m_1$ is transmitted then $r = [\sqrt{T} + n_1, n_2]$ and therefore, $P(e|m_1)$ is written as:

$$P(e|m_1) = P(n_2 - n_1 > \sqrt{T}) + P(n_1 + n_2 < -\sqrt{T})$$

Since, $n_1$, $n_2$ are zero-mean statistically independent Gaussian random variables, each with variance $\frac{N_0}{2}$, the random variables $x = n_1 - n_2$ and $y = n_1 + n_2$ are zero-mean Gaussian with variance $N_0$. Hence:

$$P(e|m_1) = \frac{1}{\sqrt{2\pi N_0}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2N_0}} dx + \frac{1}{\sqrt{2\pi N_0}} \int_{-\infty}^{-\sqrt{T}} e^{-\frac{y^2}{2N_0}} dy$$

$$= Q \left[ \sqrt{\frac{T}{N_0}} \right] + Q \left[ \sqrt{\frac{\frac{T}{N_0}}{N_0}} \right] = 2Q \left[ \sqrt{\frac{T}{N_0}} \right]$$

When $m_2$ is transmitted then $r = [n_1, n_2 + \sqrt{T}]$ and therefore:

$$P(e|m_2) = P(n_1 - n_2 > \sqrt{T}) + P(n_2 < -\sqrt{T})$$

$$= Q \left[ \sqrt{\frac{T}{N_0}} \right] + Q \left[ \sqrt{\frac{2T}{N_0}} \right]$$

Similarly from the symmetry of the problem, we obtain:

$$P(e|m_2) = P(e|m_3) = Q \left[ \sqrt{\frac{T}{N_0}} \right] + Q \left[ \sqrt{\frac{2T}{N_0}} \right]$$

Since $Q[.]$ is monotonically decreasing, we obtain:

$$Q \left[ \sqrt{\frac{2T}{N_0}} \right] < Q \left[ \sqrt{\frac{T}{N_0}} \right]$$
and therefore, the probability of error \( P(e|m_1) \) is larger than \( P(e|m_2) \) and \( P(e|m_3) \). Hence, the message \( m_1 \) is more vulnerable to errors. The reason for that is that it has both threshold lines close to it, while the other two signals have one of their threshold lines further away.

**Problem 5.25 :**

(a) If the power spectral density of the additive noise is \( S_n(f) \), then the PSD of the noise at the output of the prewhitening filter is

\[
S_\nu(f) = S_n(f)|H_p(f)|^2
\]

In order for \( S_\nu(f) \) to be flat (white noise), \( H_p(f) \) should be such that

\[
H_p(f) = \frac{1}{\sqrt{S_n(f)}}
\]

(b) Let \( h_p(t) \) be the impulse response of the prewhitening filter \( H_p(f) \). That is, \( h_p(t) = \mathcal{F}^{-1}[H_p(f)] \). Then, the input to the matched filter is the signal \( \tilde{s}(t) = s(t) * h_p(t) \). The frequency response of the filter matched to \( \tilde{s}(t) \) is

\[
\tilde{S}_m(f) = \tilde{S}^*(f)e^{-j2\pi ft_0} = S^*(f)H_p^*(f)e^{-j2\pi ft_0}
\]

where \( t_0 \) is some nominal time-delay at which we sample the filter output.

(c) The frequency response of the overall system, prewhitenig filter followed by the matched filter, is

\[
G(f) = \tilde{S}_m(f)H_p(f) = S^*(f)|H_p(f)|^2e^{-j2\pi ft_0} = \frac{S^*(f)}{S_n(f)}e^{-j2\pi ft_0}
\]

(d) The variance of the noise at the output of the generalized matched filter is

\[
\sigma^2 = \int_{-\infty}^{\infty} S_n(f)|G(f)|^2 df = \int_{-\infty}^{\infty} \frac{|S(f)|^2}{S_n(f)} df
\]

At the sampling instant \( t = t_0 = T \), the signal component at the output of the matched filter is

\[
y(T) = \int_{-\infty}^{\infty} Y(f)e^{j2\pi fT} df = \int_{-\infty}^{\infty} s(\tau)g(T - \tau) d\tau
\]

\[
= \int_{-\infty}^{\infty} S(f)\frac{S^*(f)}{S_n(f)} df = \int_{-\infty}^{\infty} \frac{|S(f)|^2}{S_n(f)} df
\]

Hence, the output SNR is

\[
\text{SNR} = \frac{y^2(T)}{\sigma^2} = \int_{-\infty}^{\infty} \frac{|S(f)|^2}{S_n(f)} df
\]
Problem 5.26:

(a) The number of bits per symbol is

\[ k = \frac{R}{R} = \frac{4800}{2400} = 2 \]

Thus, a 4-QAM constellation is used for transmission. The probability of error for an M-ary QAM system with \( M = 2^k \), is

\[ P_M = 1 - \left( 1 - 2 \left( 1 - \frac{1}{\sqrt{M}} \right) \right) Q \left[ \sqrt{\frac{3kE_b}{(M-1)N_0}} \right]^2 \]

With \( P_M = 10^{-5} \) and \( k = 2 \) we obtain

\[ Q \left[ \sqrt{\frac{2E_b}{N_0}} \right] = 5 \times 10^{-6} \implies \frac{E_b}{N_0} = 9.7682 \]

(b) If the bit rate of transmission is 9600 bps, then

\[ k = \frac{9600}{2400} = 4 \]

In this case a 16-QAM constellation is used and the probability of error is

\[ P_M = 1 - \left( 1 - 2 \left( 1 - \frac{1}{4} \right) \right) Q \left[ \sqrt{\frac{3 \times 4 \times E_b}{15 \times N_0}} \right]^2 \]

Thus,

\[ Q \left[ \sqrt{\frac{3 \times E_b}{15 \times N_0}} \right] = \frac{1}{3} \times 10^{-5} \implies \frac{E_b}{N_0} = 25.3688 \]

(c) If the bit rate of transmission is 19200 bps, then

\[ k = \frac{19200}{2400} = 8 \]

In this case a 256-QAM constellation is used and the probability of error is

\[ P_M = 1 - \left( 1 - 2 \left( 1 - \frac{1}{16} \right) \right) Q \left[ \sqrt{\frac{3 \times 8 \times E_b}{255 \times N_0}} \right]^2 \]

With \( P_M = 10^{-5} \) we obtain

\[ \frac{E_b}{N_0} = 659.8922 \]
(d) The following table gives the SNR per bit and the corresponding number of bits per symbol for the constellations used in parts a)-c).

<table>
<thead>
<tr>
<th>( k )</th>
<th>2</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>SNR (db)</td>
<td>9.89</td>
<td>14.04</td>
<td>28.19</td>
</tr>
</tbody>
</table>

As it is observed there is an increase in transmitted power of approximately 3 dB per additional bit per symbol.

**Problem 5.27 :**

Using the Pythagorean theorem for the four-phase constellation, we find:

\[
r_1^2 + r_1^2 = d^2 \implies r_1 = \frac{d}{\sqrt{2}}
\]

The radius of the 8-PSK constellation is found using the cosine rule. Thus:

\[
d^2 = r_2^2 + r_2^2 - 2r_2^2\cos(45^\circ) \implies r_2 = \frac{d}{\sqrt{2} - \sqrt{2}}
\]

The average transmitted power of the 4-PSK and the 8-PSK constellation is given by:

\[
P_{4,av} = \frac{d^2}{2}, \quad P_{8,av} = \frac{d^2}{2 - \sqrt{2}}
\]

Thus, the additional transmitted power needed by the 8-PSK signal is:

\[
P = 10\log_{10} \frac{2d^2}{(2 - \sqrt{2})d^2} = 5.3329 \text{ dB}
\]

We obtain the same results if we use the probability of error given by (see 5-2-61):

\[
P_M = 2Q\left[\sqrt{2\gamma_s \sin \frac{\pi}{M}}\right]
\]

where \( \gamma_s \) is the SNR per symbol. In this case, equal error probability for the two signaling schemes, implies that

\[
\gamma_{4,s} \sin^2 \frac{\pi}{4} = \gamma_{8,s} \sin^2 \frac{\pi}{8} \implies 10\log_{10} \frac{\gamma_{8,s}}{\gamma_{4,s}} = 20\log_{10} \frac{\sin \frac{\pi}{4}}{\sin \frac{\pi}{8}} = 5.3329 \text{ dB}
\]

Since we consider that error occur only between adjacent points, the above result is equal to the additional transmitted power we need for the 8-PSK scheme to achieve the same distance \( d \) between adjacent points.
Problem 5.28 :

For 4-phase PSK \((M = 4)\) we have the following relationship between the symbol rate \(1/T\), the required bandwidth \(W\) and the bit rate \(R = k \cdot 1/T = \frac{\log_2 M}{T}\) (see 5-2-84):

\[
W = \frac{R}{\log_2 M} \rightarrow R = W\log_2 M = 2W = 200 \text{ kbits/sec}
\]

For binary FSK \((M = 2)\) the required frequency separation is \(1/2T\) (assuming coherent receiver) and (see 5-2-86):

\[
W = \frac{M}{\log_2 M}R \rightarrow R = \frac{2W\log_2 M}{M} = W = 100 \text{ kbits/sec}
\]

Finally, for 4-frequency non-coherent FSK, the required frequency separation is \(1/T\), so the symbol rate is half that of binary coherent FSK, but since we have two bits/symbol, the bit rate is the same as in binary FSK:

\[R = W = 100 \text{ kbits/sec}\]

Problem 5.29 :

We assume that the input bits 0, 1 are mapped to the symbols -1 and 1 respectively. The terminal phase of an MSK signal at time instant \(n\) is given by

\[
\theta(n; a) = \frac{\pi}{2} \sum_{k=0}^{k} a_k + \theta_0
\]

where \(\theta_0\) is the initial phase and \(a_k\) is ±1 depending on the input bit at the time instant \(k\). The following table shows \(\theta(n; a)\) for two different values of \(\theta_0\) \((0, \pi)\), and the four input pairs of data: \(\{00, 01, 10, 11\}\).

<table>
<thead>
<tr>
<th>(\theta_0)</th>
<th>(b_0)</th>
<th>(b_1)</th>
<th>(a_0)</th>
<th>(a_1)</th>
<th>(\theta(n; a))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>(-\pi)</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>(\pi)</td>
</tr>
<tr>
<td>(\pi)</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>(\pi)</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>(\pi)</td>
</tr>
<tr>
<td>(\pi)</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>(\pi)</td>
</tr>
<tr>
<td>(\pi)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>(2\pi)</td>
</tr>
</tbody>
</table>
(ii) Based on (5-3-7), we obtain the phase states:

$$\Theta_s = \left\{ 0, \frac{3\pi}{4}, \frac{3\pi}{2}, \frac{9\pi}{4} \equiv \frac{\pi}{4}, \pi, \frac{15\pi}{4} \equiv \frac{7\pi}{4}, \frac{18\pi}{4} \equiv \frac{\pi}{2}, \frac{21\pi}{4} \equiv \frac{5\pi}{4} \right\}$$

(b)

(i) The combined states are $S_n = (\theta_n, I_{n-1}, I_{n-2})$, where $\{I_{n-1/n-2}\}$ take the values $\pm 1$. Hence there are $3 \times 2 \times 2 = 12$ combined states in all.

(ii) The combined states are $S_n = (\theta_n, I_{n-1}, I_{n-2})$, where $\{I_{n-1/n-2}\}$ take the values $\pm 1$. Hence there are $8 \times 2 \times 2 = 32$ combined states in all.

Problem 5.33:

A biorthogonal signal set with $M = 8$ signal points has vector space dimensionality 4. Hence, the detector first checks which one of the four correlation metrics is the largest in absolute value, and then decides about the two possible symbols associated with this correlation metric, based on the sign of this metric. Hence, the error probability is the probability of the union of the event $E_1$ that another correlation metric is greater in absolute value and the event $E_2$ that the signal correlation metric has the wrong sign. A union bound on the symbol error probability can be given by:

$$P_M \leq P(E_1) + P(E_2)$$

But $P(E_2)$ is simply the probability of error for an antipodal signal set: $P(E_2) = Q\left(\sqrt{\frac{E_s}{N_0}}\right)$

and the probability of the event $E_1$ can be union bounded by:

$$P(E_1) \leq 3 \left[ P(|C_2| > |C_1|) \right] = 3 \left[ 2P(C_2 > C_1) \right] = 6P(C_2 > C_1) = 6Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

where $C_i$ is the correlation metric corresponding to the $i$-th vector space dimension; the probability that a correlation metric is greater that the correct one is given by the error probability for orthogonal signals $Q\left(\sqrt{\frac{E_s}{N_0}}\right)$ (since these correlation metrics correspond to orthogonal signals). Hence:

$$P_M \leq 6Q\left(\sqrt{\frac{E_s}{N_0}}\right) + Q\left(\sqrt{\frac{2E_s}{N_0}}\right)$$

(sum of the probabilities to chose one of the 6 orthogonal, to the correct one, signal points and the probability to chose the signal point which is antipodal to the correct one).

Problem 5.34:

It is convenient to find first the probability of a correct decision. Since all signals are equiprob-
\[ P(C) = \sum_{i=1}^{M} \frac{1}{M} P(C|s_i) \]

All the \( P(C|s_i), i = 1, \ldots, M \) are identical because of the symmetry of the constellation. By translating the vector \( s_i \) to the origin we can find the probability of a correct decision, given that \( s_i \) was transmitted, as:

\[ P(C|s_i) = \int_{-\frac{d}{2}}^{\frac{d}{2}} f(n_1)dn_1 \int_{-\frac{d}{2}}^{\frac{d}{2}} f(n_2)dn_2 \ldots \int_{-\frac{d}{2}}^{\frac{d}{2}} f(n_N)dn_N \]

where the number of the integrals on the right side of the equation is \( N \), \( d \) is the minimum distance between the points and:

\[ f(n_i) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{n_i^2}{2\sigma^2}} = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{n_i^2}{N_0}} \]

Hence:

\[ P(C|s_i) = \left( \int_{-\frac{d}{2}}^{\frac{d}{2}} f(n)dn \right)^N = \left( 1 - \int_{-\infty}^{\frac{d}{2}} f(n)dn \right)^N = \left( 1 - Q \left[ \frac{d}{\sqrt{2N_0}} \right] \right)^N \]

and therefore, the probability of error is given by:

\[ P(e) = 1 - P(C) = 1 - \sum_{i=1}^{M} \frac{1}{M} \left( 1 - Q \left[ \frac{d}{\sqrt{2N_0}} \right] \right)^N = 1 - \left( 1 - Q \left[ \frac{d}{\sqrt{2N_0}} \right] \right)^N \]

Note that since:

\[ E_s = \sum_{i=1}^{N} \sum_{m,i} s_{m,i}^2 = \sum_{i=1}^{N} \left( \frac{d}{2} \right)^2 = N \frac{d^2}{4} \]

the probability of error can be written as:

\[ P(e) = 1 - \left( 1 - Q \left[ \sqrt{\frac{2E_s}{NN_0}} \right] \right)^N \]

**Problem 5.35:**

Consider first the signal:

\[ y(t) = \sum_{k=1}^{n} c_k \delta(t - kT_c) \]
where, e.g.: 
\[ p(n_{1c}) = \frac{1}{\sqrt{4\pi N_0}} \exp(-n_{1c}^2/4N_0\mathcal{E}). \]

**Problem 5.41:**

The first matched filter output is:

\[ r_1 = \int_0^T r_1(\tau) h_1(T - \tau) d\tau = \int_0^T r_1(\tau) s_{11}^* (T - (T - \tau)) d\tau = \int_0^T r_1(\tau) s_{11}^*(\tau) d\tau \]

Similarly:

\[ r_2 = \int_0^T r_1(\tau) h_2(T - \tau) d\tau = \int_0^T r_1(\tau) s_{12}^* (T - (T - \tau)) d\tau = \int_0^T r_1(\tau) s_{12}^*(\tau) d\tau \]

which are the same as those of the correlation-type receiver of Problem 5.39. From this point, following the exact same steps as in Problem 5.39, we get:

\[ r_1 = 2\mathcal{E} \cos \phi + n_{1c} + j (2\mathcal{E} \sin \phi + n_{1s}) \]
\[ r_2 = 2\mathcal{E} |\rho| \cos (\phi - a_0) + n_{2c} + j (2\mathcal{E} |\rho| \sin (\phi - a_0) + n_{2s}) \]

**Problem 5.42:**

(a) The noncoherent envelope detector for the on-off keying signal is depicted in the next figure.

(b) If \( s_0(t) \) is sent, then the received signal is \( r(t) = n(t) \) and therefore the sampled outputs \( r_c, r_s \) are zero-mean independent Gaussian random variables with variance \( N_0/2 \). Hence, the random variable \( r = \sqrt{r_c^2 + r_s^2} \) is Rayleigh distributed and the PDF is given by:

\[ p(r|s_0(t)) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \]

\[ = \frac{2r}{N_0} e^{-\frac{r^2}{N_0}} \]

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If \( s_1(t) \) is transmitted, then the received signal is:
\[
r(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t + \phi) + n(t)
\]

Crosscorrelating \( r(t) \) by \( \sqrt{\frac{2}{T}} \cos(2\pi f_c t) \) and sampling the output at \( t = T \), results in
\[
r_c = \int_0^T r(t) \sqrt{\frac{2}{T}} \cos(2\pi f_c t) dt
\]
\[
= \int_0^T \frac{2\sqrt{E_b}}{T_b} \cos(2\pi f_c t + \phi) \cos(2\pi f_c t) dt + \int_0^T n(t) \sqrt{\frac{2}{T}} \cos(2\pi f_c t) dt
\]
\[
= \frac{2\sqrt{E_b}}{T_b} \int_0^T \frac{1}{2} (\cos(2\pi f_c t + \phi) + \cos(\phi)) dt + n_c
\]
\[
= \sqrt{E_b} \cos(\phi) + n_c
\]

where \( n_c \) is zero-mean Gaussian random variable with variance \( \frac{N_0}{2} \). Similarly, for the quadrature component we have:
\[
r_s = \sqrt{E_b} \sin(\phi) + n_s
\]

The PDF of the random variable \( r = \sqrt{r_c^2 + r_s^2} = \sqrt{E_b + n_c^2 + n_s^2} \) follows the Rician distribution:
\[
p(r|s_1(t)) = \frac{r}{\sigma^2} e^{-\frac{r^2 + \xi_c^2}{2\sigma^2}} I_0 \left( \frac{r\sqrt{E_b}}{\sigma} \right) = \frac{2r}{N_0} e^{-\frac{r^2 + \xi_c^2}{2N_0}} I_0 \left( \frac{2r\sqrt{E_b}}{N_0} \right)
\]

(c) For equiprobable signals the probability of error is given by:
\[
P(\text{error}) = \frac{1}{2} \int_{-\infty}^{v_T} p(r|s_1(t)) dr + \frac{1}{2} \int_{v_T}^{\infty} p(r|s_0(t)) dr
\]

Since \( r > 0 \) the expression for the probability of error takes the form
\[
P(\text{error}) = \frac{1}{2} \int_0^{v_T} p(r|s_1(t)) dr + \frac{1}{2} \int_{v_T}^{\infty} p(r|s_0(t)) dr
\]
\[
= \frac{1}{2} \int_0^{v_T} \frac{r}{\sigma^2} e^{-\frac{r^2 + \xi_c^2}{2\sigma^2}} I_0 \left( \frac{r\sqrt{E_b}}{\sigma} \right) dr + \frac{1}{2} \int_{v_T}^{\infty} \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr
\]

The optimum threshold level is the value of \( v_T \) that minimizes the probability of error. However, when \( \frac{E_b}{N_0} \gg 1 \) the optimum value is close to: \( \frac{\sqrt{2E_b}}{2} \) and we will use this threshold to simplify the analysis. The integral involving the Bessel function cannot be evaluated in closed form. Instead of \( I_0(x) \) we will use the approximation:
\[
I_0(x) \approx \frac{e^x}{\sqrt{2\pi x}}
\]
which is valid for large $x$, that is for high SNR. In this case:

$$\frac{1}{2} \int_0^{\sqrt{E_b}} \frac{r}{\sigma^2} e^{-\frac{r^2+\sigma^2}{2\sigma^2}} I_0 \left( \frac{r}{\sigma^2} \right) dr \approx \frac{1}{2} \int_0^{\sqrt{E_b}} \frac{r}{2\pi\sigma^2\sqrt{E_b}} e^{-\frac{(r-\sqrt{E_b})^2}{2\sigma^2}} dr$$

This integral is further simplified if we observe that for high SNR, the integrand is dominant in the vicinity of $\sqrt{E_b}$ and therefore, the lower limit can be substituted by $-\infty$. Also

$$\sqrt{\frac{r}{2\pi\sigma^2\sqrt{E_b}}} \approx \sqrt{\frac{1}{2\pi\sigma^2}}$$

and therefore:

$$\frac{1}{2} \int_0^{\sqrt{E_b}} \sqrt{\frac{r}{2\pi\sigma^2\sqrt{E_b}}} e^{-\frac{(r-\sqrt{E_b})^2}{2\sigma^2}} dr \approx \frac{1}{2} \int_{-\infty}^{\sqrt{E_b}} \sqrt{\frac{1}{2\pi\sigma^2}} e^{-\frac{(r-\sqrt{E_b})^2}{2\sigma^2}} dr = \frac{1}{2} Q \left[ \sqrt{\frac{E_b}{2N_0}} \right]$$

Finally:

$$P(\text{error}) = \frac{1}{2} Q \left[ \sqrt{\frac{E_b}{2N_0}} \right] + \frac{1}{2} \int_{-\infty}^{\sqrt{E_b}} \frac{2r}{\sqrt{2\pi}} e^{-\frac{r^2}{2\sigma^2}} dr \leq \frac{1}{2} Q \left[ \sqrt{\frac{E_b}{2N_0}} \right] + \frac{1}{2} e^{-\frac{E_b}{4N_0}}$$

Problem 5.43:

(a) $D = Re \left( V_m^*V_{m-1} \right)$ where $V_m = X_m + jY_m$. Then:

$$D = Re \left( (X_m + jY_m)(X_{m-1} - jY_{m-1}) \right)$$

$$= X_mX_{m-1} + Y_mY_{m-1}$$

$$= \left( \frac{X_m + X_{m-1}}{2} \right)^2 - \left( \frac{X_m - X_{m-1}}{2} \right)^2 + \left( \frac{Y_m + Y_{m-1}}{2} \right)^2 - \left( \frac{Y_m - Y_{m-1}}{2} \right)^2$$

(b) $V_k = X_k + jY_k = 2aE \cos(\theta - \phi) + j2aE \sin(\theta - \phi) + N_{k,real} + N_{k,imag}$. Hence:

$$U_1 = \frac{X_m + X_{m-1}}{2}, \quad E(U_1) = 2aE \cos(\theta - \phi)$$

$$U_2 = \frac{Y_m + Y_{m-1}}{2}, \quad E(U_2) = 2aE \sin(\theta - \phi)$$

$$U_3 = \frac{X_m - X_{m-1}}{2}, \quad E(U_3) = 0$$

$$U_4 = \frac{Y_m - Y_{m-1}}{2}, \quad E(U_4) = 0$$

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