Problem 6.1 (ECE 515M 7-11)

(a) This is very similar to the orthogonal signaling case covered in class.

\[ \gamma(t) = e^{j\theta} A[n] u(t) + w(t) \]

On basis \( \Phi_0(t) = \frac{1}{\| h \|} h(t) \); pick any \( \Phi_n(t) \) to get complete basis.

\[ \langle \gamma(t), \Phi_n(t) \rangle = e^{j\theta} A[n] \langle h, \Phi_n \rangle + \langle w, \Phi_n \rangle \]

\[ = A[n] e^{j\theta} \| h \| \delta[n] + \text{w}[n] \]

Since \( \{\Phi_n\} \) is orthonormal set \( \text{w}[n] \) is an IID zero-mean.

\[ \rho_n = \text{variance} \text{ c.s. symmetric} \]

Hence, since \( w(t) \) is independent of \( A[n] \), \( \text{w}[n] \) is independent of \( A[n] e^{j\theta} \) and since \( \text{w}[n] \) is also IID, \( \{x[n] e^{j\theta}\} \) is independent of \( A[n] \) \& \( x[n] \). Hence, \( x[n] \) suffices for optimal detection.

ie. the optimal detector decides on \( A[n] \) based on

\[ x[n] e^{j\theta} \]

\[ \text{c.s. symmetric} \]

\[ A_{\text{max}} = A_{\text{max}} = \text{argmax} \sum_{k=0}^{12} f_{x[n] 1/k} (x[n] 1/k) \]

\[ = \text{argmax} \int_0^{2\pi} f_{\theta}[x[n] 1/k, \theta] f_{\theta}(\theta) \, d\theta \]
But \( \sum_{n=1}^{\infty} e^{j\theta} (A[n]) \frac{1}{1+n} \) = \( C e^{-\frac{1}{m^2} \| \mathbf{x} \|^2 - 2 e^{j\theta} \mathbf{y} \| \mathbf{y} \|^2} \)

\( C' e^{-\frac{1}{m^2} \| \mathbf{y} \|^2} e^{-2 \frac{e^{j\theta} \mathbf{y} \| \mathbf{y} \|^2} \| \mathbf{y} \|^2} \)

which gives

\[ \hat{A}_{\text{MAX}} = \max_{k \in \mathbb{C}} I_0 \left( \frac{2}{\omega} \sqrt{\| \mathbf{x} \|^2}, \frac{\| \mathbf{y} \|^2}{\omega} \right) e^{-\frac{\| \mathbf{y} \|^2}{\omega^2}} \]

So

\[ I_0 (0) \leq I_0 \left( \frac{2}{\omega} \sqrt{\| \mathbf{x} \|^2}, \frac{\| \mathbf{y} \|^2}{\omega} \right) e^{-\frac{\| \mathbf{y} \|^2}{\omega^2}} \]

\[ \Rightarrow \hat{A} = 1 \quad \hat{A}_0 = 0 \]

\[ \Rightarrow I_0 (0) \leq I_0 \left( \frac{2}{\omega} \sqrt{\| \mathbf{x} \|^2}, \frac{\| \mathbf{y} \|^2}{\omega} \right) e^{-\frac{\| \mathbf{y} \|^2}{\omega^2}} \]

So the test reduces to:

\[ \text{Let } F(\gamma) = \frac{\hat{A}_0 \left( I_0 \left( \frac{k}{\omega} \sqrt{\| \mathbf{x} \|^2}, \frac{\| \mathbf{y} \|^2}{\omega} \right) \right)}{\ln (I_0 (\gamma))} \Rightarrow F(\gamma) = \frac{\hat{A}_0 \left( I_0 \left( \frac{k}{\omega} \sqrt{\| \mathbf{x} \|^2}, \frac{\| \mathbf{y} \|^2}{\omega} \right) \right)}{\ln (I_0 (\gamma))} \]

Then the test reduces to:

\[ \int_{-\infty}^{\infty} \left| \mathbf{x}(t) \right|^2 |h(t)|^2 dt \]

\[ \Rightarrow \mathbf{y} = \mathbf{\text{ const }} \]

\[ \Rightarrow \gamma = \frac{\pi^2}{2} F \left( \frac{\mathbf{\text{ const }}}{\omega^2} \right) \]

(b) Let \( A_k = |\mathbf{p}| e^{j\phi_k} \) denote the \( k \)-th constellation symbol in polar form,

\[ |\mathbf{p}| > 0 \]

Since discrimination relies on all \( A_k \) having distinct magnitudes, if \( |\mathbf{p}| = \) for some \( n \) then discrimination is not possible, even in the absence of noise. Examples include PSK & QAM constellations. So to be able to discriminate \( W \), \( |\mathbf{p}| \rightarrow 0 \) and \( \omega \rightarrow 0 \), all constellation symbols must have different magnitudes. In
\( (c) \quad p_Y(y) = \frac{1}{2} pr(H = 0 | H = 0) + \frac{1}{2} pr(H = 1 | H = 0) \)

Let \( L = \int s\gamma(t)\xi(t)\,dt \)

Given \( H = 0 \), \( L = \sqrt{\int s\gamma(t)\xi(t)\,dt} = |\varepsilon| \)

\( m \sim \mathcal{N}(0, \sigma^2||\varepsilon||^2) \)

So \( p_Y(\varepsilon | H = 0) = p_Y(H = 1 | H = 0) = p_Y(|\varepsilon| > \gamma) \)

Given \( H = 1 \), \( L = \sqrt{e^{\theta}||\varepsilon||^2 + |\varepsilon|} \)

\( m \sim \mathcal{N}(0, \sigma^2||\varepsilon||^2) \)

So \( p_Y(\varepsilon | H = 1) = p_Y(H = 0 | H = 1) = p_Y(e^{\theta}||\varepsilon||^2 + |\varepsilon| < \gamma) \)

where \( \gamma = \frac{\sigma^2}{2} \left( \frac{\int s^{2}\gamma(t)^2\,dt}{\sigma^2} \right) \)
Problem 6.2

(a) \( N = 2, \quad 2\pi N v = 2\pi mk \Rightarrow \frac{2\pi}{T} = \frac{2\pi mk}{T} \quad k = 1, 2. \)

Without loss of generality assume \( k_2 > k_1 \)

\[ s_{c_0} = \frac{1}{2}(s_{c_1} + s_{c_2}) = \frac{\pi}{T} \left[ m_1 + m_2 \right] / 2. \]

\[ s_{c_k} = \frac{1}{s_0} \left( s_{c_2} - s_{c_1} \right) = \left( \frac{m_2 - m_1}{2} \right) \frac{2\pi}{T} \]

\[ s_{c_1} = s_{c_e} - s_{c_d}, \quad s_{c_2} = s_{c_e} + s_{c_d}, \quad b_k = \lambda^{-1} \quad u = 1 \]

So \( b_{\text{A}1} = b_{\text{A}2} + b_{\text{A}3} \quad b_{\text{A}1} = b_{\text{A}2} \quad b_{\text{A}3} \)

Hence \( g_{\text{A}1}(t) = \sin \left( s_{c_e} t + \frac{2\pi}{T} \right) \]

Hence we need to pick \( d(t) \) so \( \theta(0) \) is \( \sin \) and \( \theta(t) \) be \( \cos \) in (1) can be expressed as in (2).

Let \( s_1(t) \) denote the signal in (1) and \( s_2(t) \) the signal in (2). Need to show that \( n > 0 \)

For \( N \leq t \leq (N+1)T \) \( s_1(t) = s_2(t) \) for proper choice of \( d(t) \).

First, since CPSK is memoryless, \( d(t) = 0 \) for \( t < 0 \) and \( t > T \).

Next, \( n = 0 \), \( t = 0 \) \( s_1(t) = \theta \) \( \Rightarrow \) \( s_2(0) = \sin(\theta(0)) \Rightarrow \theta(0) = 0 \pi \)

Also \( n = 0 \), \( t = 0 \) \( s_1(t) > 0 \) \( \Rightarrow \) \( \theta(0) = 2\pi \). Pick \( \theta(0) = 0 \).

Finally, need to pick \( d(t) : \)

\[ \text{For } N \leq t \leq (N+1)T \quad s_1(t) = s_2(t) \Rightarrow \]

\[ g_{\text{A}1}(t - N) = \sin \left( s_{c_e} t + \int_0^T u(t) dt \right) \]
Clearly, since the phase in the left hand side of (**) varies linearly with t, to get the RHS to vary linearly with t it is necessary that \( \int_0^t u(t) \, dt \) is a linear function of t, which do to (**) implies that \( d(t) \) is a constant in \([0,T]\), i.e.

\[
\frac{c}{d(t)}
\]

Equating the rates of change on both sides of (**) gives

\[
S_e + \sum_d b_d [e] = S_e + \sum_d c b_d [e] \quad \Rightarrow \quad c = 1
\]

Noting also that

\[
S_e t = S_e (t - n T) + \sum_{k=0}^{n-1} S_e (k+1) T - k T
\]

\[
= S_e (t - n T) + \sum_{k=0}^{n-1} \int_{k T}^{(k+1) T} d(t) \, dt
\]

and substituting in (**) we get

\[
\sin(S_e (t - n T) + \sum_d b_d [e] (t - n T)) = \sin(S_e (t + n T)) + \sum_{k=0}^{n-1} \left[ \frac{S_e + \sum_d b_d [e]}{k T} \right] \left[ \int_{k T}^{(k+1) T} d(t) \, dt + \sum_d b_d [e] (t - n T) \right] \int_{2\pi}^1 \text{Integrate term outside}
\]

\[
\sin(S_e (t - n T) + \sum_d b_d [e] (t - n T)) = \sin(S_e (t - n T) + \sum_d b_d [e] (t - n T))
\]
(b) Again assume \( S_1 < S_2 < S_3 < \cdots < S_n \), and let 

\[ S_d = \frac{S_1 + S_2}{2}, \quad S_{d-1} = \frac{S_{n-1} + S_n}{2} \]

Then, while 

\[ S_k = S_c + b_k S_d, \quad \text{i.e.} \quad S_{AM-k} = S_c + b_k S_d, \]

Note that \( |b_k| = 1 \) \( \Rightarrow \) \( b_1 < b_2 < \cdots < b_n = 1 \), 

i.e. we have \( AM \), 

\( u(t) = \sum_{k=0}^{\infty} b_k S_d e^{(t-kT)} \)

CPFSK \( \iff \) \( S_{d+kT} = 2\pi m, \quad m \in \mathbb{Z}, \quad k \neq 0, \quad k \neq 1, 2, \ldots, N^2 \).

The proof is then identical to that of part (a), since after eqn 49, all steps remain identical.