II. F Exercises

1. Find the minimum Bayes risk for the binary channel of Example II. B.1.

2. Suppose $Y$ is a random variable that, under hypothesis $H_0$, has pdf

$$p_0(y) = \begin{cases} (2/3)(y+1), & 0 \leq y \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

and, under hypothesis $H_1$, has pdf

$$p_1(y) = \begin{cases} 1, & 0 \leq y \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the Bayes rule and minimum Bayes risk for testing $H_0$ versus $H_1$ with uniform costs and equal priors.

(b) Find the minimax rule and minimax risk for uniform costs.

(c) Find the Neyman-Pearson rule and the corresponding detection probability for false-alarm probability $\alpha \in (0, 1)$.

3. Repeat Exercise 2 for the situation in which $p_j$ is given instead by

$$p_j(y) = \frac{(j+1)}{2} e^{-(j+1)|y|}, y \in \mathbb{R}, j = 0, 1.$$ 

For parts (a) and (b) assume costs

$$C_{ij} = \begin{cases} 0, & \text{if } i = j \\ 1, & \text{if } i = 1 \text{ and } j = 0 \\ 3/4, & \text{if } i = 0 \text{ and } j = 1, \end{cases}$$

and for part (a) assume priors $\pi_0 = 1/4$ and $\pi_1 = 3/4$.

4. Repeat Exercise 2 for the situation in which $p_0$ and $p_1$ are given instead by

$$p_0(y) = \begin{cases} e^{-y}, & y \geq 0 \\ 0, & y < 0 \end{cases}$$

and

$$p_1(y) = \begin{cases} \sqrt{2/\pi} e^{-y^2/2}, & y \geq 0 \\ 0, & y < 0. \end{cases}$$

For part (a) consider arbitrary priors.
5. Repeat Exercise 2 for the hypothesis pair

\[ H_0 : Y \text{ has density } p_0(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}, y \in \mathbb{R} \]

versus

\[ H_1 : Y \text{ has density } p_1(y) = \begin{cases} 
1/5, & \text{if } y \in [0,5] \\
0, & \text{if } y \notin [0,5].
\end{cases} \]

For part (a) assume priors \( \pi_0 = 3/4 \) and \( \pi_1 = 1/4 \).

6. Repeat Exercise 2 for the hypothesis pair

\[ H_0 : Y = N - s \]

versus

\[ H_1 : Y = N + s \]

where \( s > 0 \) is a fixed real number and \( N \) is a continuous random variable with density

\[ p_N(n) = \frac{1}{\pi(1 + n^2)}, \; n \in \mathbb{R}. \]

7. (a) Consider the hypothesis pair

\[ H_0 : Y = N \]

versus

\[ H_1 : Y = N + S \]

where \( N \) and \( S \) are independent random variables each having pdf

\[ p(x) = \begin{cases} 
e^{-x}, & x \geq 0 \\
0, & x < 0.
\end{cases} \]

Find the likelihood ratio between \( H_0 \) and \( H_1 \).

(b) Find the threshold and detection probability for \( \alpha \)-level Neyman-Pearson testing in (a).

(c) Consider the hypothesis pair

\[ H_0 : Y_k = N_k, \; k = 1, \ldots, n \]

versus

\[ H_1 : Y_k = N_k + S, \; k = 1, \ldots, n \]

where \( n > 1 \) and \( N_1, \ldots, N_n \), and \( S \) are independent random variables each having the pdf given in (a). Find the likelihood ratio.

(d) Find the threshold for \( \alpha \)-level Neyman-Pearson testing in (c).
8. Show that the minimum-Bayes-risk function \( V \) (defined in Section II.C) is concave and continuous in \([0, 1]\). [After showing that \( V \) is concave you may use the fact that any concave function on \([0, 1]\) is continuous on \((0, 1)\).]

9. Suppose we have a real observation \( Y \) and binary hypotheses described by the following pair of pdf's:

\[
p_0(y) = \begin{cases} 
(1 - |y|), & \text{if } |y| \leq 1 \\
0, & \text{if } |y| > 1
\end{cases}
\]

and

\[
p_1(y) = \begin{cases} 
(2 - |y|)/4, & \text{if } |y| \leq 2 \\
0, & \text{if } |y| > 2
\end{cases}
\]

(a) Assume that the costs are given by

\[
C_{01} = 2C_{10} > 0, \\
C_{00} = C_{11} = 0.
\]

Find the minimax test of \( H_0 \) versus \( H_1 \) and the corresponding minimax risk.

(b) Find the Neyman-Pearson test of \( H_0 \) versus \( H_1 \) with false-alarm probability \( \alpha \). Find the corresponding power of the test.

10. Suppose we observe a random variable \( Y \) given by

\[
Y = N + \theta \lambda
\]

where \( \theta \) is either 0 or 1, \( \lambda \) is a fixed number between 0 and 2, and where \( N \) is a random variable that has a uniform density on the interval \((-1, 1)\). We wish to decide between the hypotheses

\[
H_0 : \theta = 0 \\
versus \\
H_1 : \theta = 1.
\]

(a) Find the Neyman-Pearson decision rule for false-alarm probability ranging from 0 to 1.

(b) Find the power of the Neyman-Pearson decision rule as a function of the false-alarm probability and the parameter \( \lambda \). Sketch the receiver operating characteristics.

11. Consider the simple hypothesis testing problem for the real-valued observation \( Y \):

\[
H_0 : p_0(y) = \exp(-y^2/2)/\sqrt{2\pi}, \quad y \in \mathbb{R}
\]

\[
H_1 : p_1(y) = \exp(-(y - 1)^2/2)/\sqrt{2\pi}, \quad y \in \mathbb{R}
\]
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Suppose the cost assignment is given by \( C_{00} = C_{11} = 0, C_{10} = 1, \) and \( C_{01} = N. \) Investigate the behavior of the Bayes rule and risk for equally likely hypotheses and the minimax rule and risk when \( N \) is very large.

12. Consider a simple binary hypothesis testing problem. For a decision rule \( \delta, \) denote the false-alarm and miss probabilities by \( P_F(\delta) \) and \( P_M(\delta), \) respectively. Consider the performance measure:

\[
\rho(\delta) \begin{align*}
\triangleq [P_F(\delta)]^2 + [P_M(\delta)]^2;
\end{align*}
\]

and let \( \delta_0 \) denote a decision rule minimizing \( \rho(\delta) \) over all randomized decision rules \( \delta. \)

(a) Show that \( \delta_0 \) must be a likelihood-ratio test.
(b) For \( \pi_0 \in [0,1], \) define the function \( V \) by

\[
V(\pi_0) = \min_{\delta} \{ \pi_0 P_F + (1 - \pi_0) P_M \}.
\]

Suppose that \( V(\pi_0) \) achieves its maximum on \([0,1]\) at the point \( \pi_0 = 1/2. \) Show that \( \delta_0 \) is a Bayes rule for prior \( \pi_0 = 1/2. \) [Hint: Note that we can write \( 2\rho(\delta) = [P_F(\delta) + P_M(\delta)]^2 + [P_F(\delta) - P_M(\delta)]^2 \cdot \]

13. Consider the following Bayes decision problem: The conditional density of the real observation \( Y \) given the real parameter \( \Theta = \theta \) is given by

\[
p_\theta(y) = \begin{cases} \theta e^{-\theta y}, & y \geq 0 \\ 0, & y < 0. \end{cases}
\]

\( \Theta \) is random variable with density

\[
w(\theta) = \begin{cases} \alpha e^{-\alpha \theta}, & \theta \geq 0 \\ 0, & \theta < 0. \end{cases}
\]

where \( \alpha > 0. \) Find the Bayes rule and minimum Bayes risk for the hypotheses

\( H_0 : \Theta \in (0, \beta) \) versus \( H_1 : \Theta \in [\beta, \infty) \)

where \( \beta > 0 \) is fixed. Assume the cost structure

\[
C[i, \theta] = \begin{cases} 1, & \text{if } \theta \notin \Lambda_i \\ 0, & \text{if } \theta \in \Lambda_i. \end{cases}
\]
by $C_{06} = C_{11} = 0, C_{10} = 1$, of the Bayes rule and risk for imax rule and risk when $N$ is

ing problem. For a decision
ss probabilities by $P_F(\delta)$ and
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$|P_M(\delta)|^2$;

sizing $\rho(\delta)$ over all randomized
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$\pi \ln V$ by

$= (1 - \pi_0)P_M].$

maximum on $[0,1]$ at the point
eyes rule for prior $\pi_0 = 1/2$.

$2\rho(\delta) = [P_F(\delta) + P_M(\delta)]^2 +$

problem: The conditional den-
e real parameter $\Theta = \theta$ is given

$y \geq 0$

$y < 0.$

$\theta \geq 0$

$\theta < 0.$

id minimum Bayes risk for the

$(0, \beta)^{\Delta} \Lambda_0$

$[\beta, \infty)^{\Delta} \Lambda_1$

: structure

if $\theta \notin \Lambda_1$

if $\theta \in \Lambda_1.$

14. Repeat Exercise 13 for the case in which $Y$ consists of $n$ inde-

dependent (conditioned on $\Theta$) and identically distributed observations

$Y = Y_1, \ldots, Y_n$ each with the conditional density given in 13. You

need not find the Bayes risk in closed form.

15. Consider the composite hypothesis testing problem:

$H_0 : Y \text{ has density } p_0(y) = \frac{1}{2} e^{-|y|}, \ y \in \mathbb{R}$

versus

$H_1 : Y \text{ has density } p_0(y) = \frac{1}{2} e^{-|y| - \theta}, \ y \in \mathbb{R}, \theta > 0.$

(a) Describe the locally most powerful $\alpha$-level test and derive its

defined function.

(b) Does a uniformly most powerful test exist? If so, find it and

derive its power function. If not, find the generalized likelihood

ratio test for $H_0$ versus $H_1.$

16. In Section 2, we formulated and solved the binary Bayesian hypo-
tesing problem. Generalize this formulation and solution to $M$ hy-

theses for $M > 2.$

17. Formulate the $M$-ary minimax hypothesis-testing problem. Show that

a Bayes equalizer rule (if one exists) is minimax.

18. How would you formulate a criterion analogous to the Neyman-


19. Consider the following pair of hypotheses concerning a sequence

$Y_1, Y_2, \ldots, Y_n$ of independent random variables

$H_0 : Y_k \sim N(\mu_0, \sigma_0^2), \ k = 1, 2, \ldots, n$

versus

$H_1 : Y_k \sim N(\mu_1, \sigma_1^2), \ k = 1, 2, \ldots, n$

where $\mu_0, \mu_1, \sigma_0^2,$ and $\sigma_1^2$ are known constants.

(a) Show that the likelihood ratio can be expressed as a function of

the parameters $\mu_0, \mu_1, \sigma_0^2,$ and $\sigma_1^2,$ and the quantities $\sum_{k=1}^{n} Y_k^2$

and $\sum_{k=1}^{n} Y_k.$

(b) Describe the Neyman-Pearson test for the two cases ($\mu_0 =

\mu_1, \sigma_1^2 > \sigma_0^2$) and ($\sigma_1^2 = \sigma_0^2, \mu_1 > \mu_0$).

(c) Find the threshold and ROC's for the case $\mu_0 = \mu_1, \sigma_1^2 > \sigma_0^2$

with $n = 1.$

20. Consider the hypotheses of Exercise 19 with $\mu_\Delta = \mu_1 > \mu_0 = 0$ and

$\sigma_\Delta = \sigma_1^2 = \sigma_0^2 > 0.$ Does there exist a uniformly most powerful test
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of these hypotheses under the assumption that \( \mu \) is known and \( \sigma^2 \) is not? If so, find it and show that it is UMP. If not, show why and find the generalized likelihood ratio test.

21. Suppose \( Y_1, Y_2, \ldots, Y_n \) is a sequence of random observations, each taking the values 0 and 1 with probabilities 1/2. Consider the following two hypotheses concerning \( Y_1, Y_2, \ldots, Y_n \):

\[
H_0 : Y_1, Y_2, \ldots, Y_n \text{ are independent}
\]

\[
H_1 : p_1(y_k|y_1, y_2, \ldots, y_{k-1}) = \begin{cases} 
3/4 & \text{if } y_k = y_{k-1} \\
1/4 & \text{if } y_k \neq y_{k-1}
\end{cases}, \quad k = 2, 3, \ldots, n,
\]

where \( p_1(y_k|y_1, y_2, \ldots, y_{k-1}) \) denotes the conditional probability that \( Y_k = y_k \) given that \( Y_1 = y_1, Y_2 = y_2, \ldots, Y_{k-1} = y_{k-1} \). Find the Bayes decision rule for testing \( H_0 \) versus \( H_1 \) under the assumption of uniform costs and equal priors.