ESTIMATION AND DETECTION THEORY

HOMEWORK # 8:

Please work out the ten (10) problems stated below – HVP refers to the text: H. Vincent Poor, *An Introduction to Signal Detection and Estimation* (Second Edition), Springer Texts in Electrical Engineering Springer, New York (NY), 2010. With this in mind, Exercise II.2 (HVP) refers to Exercise 2 for Chapter II of HVP. Exercises are located at the end of each chapter.

Show work and explain reasoning.

1. Solve Exercise IV.5 (HVP).

2. Assume the family \( \{F_\theta, \theta \in \Theta\} \) to be an exponential family (with respect to some distribution function \( F \) on \( \mathbb{R}^k \)) with density functions of the form

\[
f_\theta(y) = C(\theta)q(y)e^{Q(\theta)'K(y)F - a},
\]

for every \( \theta \) in \( \Theta \) with Borel mappings \( C : \Theta \to \mathbb{R}_+, Q : \Theta \to \mathbb{R}^q, q : \mathbb{R}^k \to \mathbb{R}_+, \) and \( K : \mathbb{R}^k \to \mathbb{R}^q \).

2.a Specialize Conditions (CR1)–(CR5) in terms (of properties) of the Borel mappings entering the definition of the exponential family. In particular, show that the regularity condition (CR5) is equivalent to

\[
\left( \frac{\partial}{\partial \theta_i} Q(\theta) \right)' \mathbb{E}_\theta [K(Y)] = - \frac{\partial}{\partial \theta_i} \log C(\theta), \quad \theta_i \in \Theta, \quad i = 1, \ldots, p.
\]

2.b Find an expression for the Fisher information matrix \( M(\theta) \) in terms of the covariance \( \text{Cov}_\theta [K(Y)] \). **HINT:** Use Part 2.a together with the fact that

\[
\frac{\partial}{\partial \theta_i} \log f_\theta(y) = \frac{\partial}{\partial \theta_i} \log C(\theta) + \frac{\partial}{\partial \theta_i} Q(\theta)'K(y), \quad i = 1, \ldots, p, \quad y \in S.
\]

2.c What are the conditions that need to hold for an estimator \( g : \mathbb{R}^k \to \mathbb{R}^p \) to be a regular estimator?

4. Solve Exercise IV.21 Part (a) (HVP).

5. Solve Exercise IV.22 (HVP).

6. In the context of the Cramér-Rao bounds, with arbitrary $p$, consider the $\mathbb{R}^p$-valued rv $U(\theta, Y)$ given by

$$U(\theta, Y) = g(Y) - \theta - b_\theta(g) - (I_p + \nabla_\theta b_\theta(g)) M(\theta)^{-1} \nabla_\theta \log f_\theta(Y), \quad \theta \in \Theta.$$ 

Show that the rv $U(\theta, Y)$ has zero mean and that its covariance matrix is given by

$$\text{Cov}_\theta[U(\theta, Y)] = \Sigma_\theta(g) - b_\theta(g)b_\theta(g)' - (I_p + \nabla_\theta b_\theta(g)) M(\theta)^{-1} (I_p + \nabla_\theta b_\theta(g))'.$$  \hspace{1cm} (1.1)

Use this fact to show that the Cramér-Rao bound is equivalent to the statement that the covariance matrix $\text{Cov}_\theta[U(\theta, Y)]$ is positive semi-definite. Explore what happens when the bound is achieved.

In Exercises 7 to 9, a family of distributions $\{F_\theta, \theta \in \Theta\}$ is given. For each $n = 1, 2, \ldots,$ let $\{F_\theta^{(n)}, \theta \in \Theta\}$ denote the corresponding families associated with $n$ i.i.d. samples. In each case, (i) compute the Fisher information matrices $M^{(n)}(\theta)$ for each $\theta$ in $\Theta$; (ii) find the efficient estimator for $\theta$ on the basis of the samples $Y_1, \ldots, Y_n$; (iii) find the ML estimator of $\theta$ on the samples $Y_1, \ldots, Y_n$; (iv) Are these estimators unbiased? asymptotically unbiased? consistent? asymptotically normal?

7. Here $\Theta = (0, \infty)$ and for each $\theta > 0$, $F_\theta$ is an exponential distribution with parameter $\theta$, namely

$$F_\theta(y) = \left(1 - e^{-\theta y}\right)^+, \quad y \in \mathbb{R}.$$ 

8. Here $\Theta = \mathbb{R}$ and for each $\theta > 0$, $F_\theta$ is the shifted Cauchy distribution whose probability density function is given by

$$f_\theta(y) = \frac{1}{\pi(1 + (y - \theta)^2)}, \quad y \in \mathbb{R}.$$ 

9. Here $\Theta = (0, \infty)$ and for each $\theta > 0$, $F_\theta$ is the Gaussian distribution with zero mean and variance $\theta$.

10. Solve Exercise IV.23 (HVP).