ESTIMATION AND DETECTION THEORY

HOMEWORK # 2:

Please work out the nine (9) problems stated below – HVP refers to the text: H. Vincent Poor, An Introduction to Signal Detection and Estimation (Second Edition), Springer Texts in Electrical Engineering Springer, New York (NY), 2010. With this in mind, Exercise II.2 (HVP) refers to Exercise 2 for Chapter II of HVP. Exercises are located at the end of each chapter.

Show work and explain reasoning.

1. Solve Part (a) of Exercise II.5 (HVP).

2. Solve Part (a) of Exercise II.6 (HVP).

3. Solve Part (a) of Exercise II.7 (HVP)

4. Recall that a rv $Z$ is said to be Rayleigh distributed with parameter $\sigma^2 > 0$ if its probability distribution $F_{\sigma^2}$ admits a probability density function $f_{\sigma^2} : \mathbb{R} \to \mathbb{R}_+$ given by

\[
  f_{\sigma^2}(z) = \begin{cases} 
  0 & \text{if } z < 0 \\
  \frac{z}{\sigma^2} e^{-\frac{z^2}{2\sigma^2}} & \text{if } z \geq 0.
\end{cases}
\]

What are the likelihood ratio tests for the binary hypothesis testing problem

$$H_1 : \quad Y_1, \ldots, Y_k \text{ i.i.d. with } Y_\ell \sim F_{\sigma_1^2}, \ \ell = 1, \ldots, k$$

$$H_0 : \quad Y_1, \ldots, Y_k \text{ i.i.d. with } Y_\ell \sim F_{\sigma_0^2}, \ \ell = 1, \ldots, k$$

with $\sigma_0^2 \neq \sigma_1^2$, both strictly positive?

5. In Problem 4 try to compute the probabilities $P_F(L_{rt, \eta})$ and $P_M(L_{rt, \eta})$ with $\eta > 0$. [HINT: What is the distribution of the rv $Z^2$ when $Z$ is Rayleigh distributed with parameter $\sigma^2 > 0$?]
6. You are facing the binary hypothesis testing problem

\[ H_1 : \ Y \sim N(1, \sigma^2) \]
\[ H_0 : \ Y \sim N(0, \sigma^2) \]

with \( \sigma^2 > 0 \) and uniform prior on \( H \). However, as you start thinking about its solution, you are told that the measurement \( Y \) is not available, and that you will have access only to \( Z = Y^2 \). Based on this modified measurement, obtain a decision rule which minimizes the probability of error criterion.

7. In the context of Problem 6, explore the loss of performance from using measurement \( Z \) instead of the original measurement \( Y \).

8. A rv \( Y \) is said to be a Bernoulli rv with parameter \( a \) (in \( [0, 1] \)) if

\[ \mathbb{P}[Y = 1] = a \quad \text{and} \quad \mathbb{P}[Y = 0] = 1 - a. \]

Consider now the binary hypothesis testing problem

\[ H_1 : \ Y_1, \ldots, Y_k \ i.i.d. \ \text{with} \ Y_\ell \sim \text{Ber}(a_1), \ \ell = 1, \ldots, k \]
\[ H_0 : \ Y_1, \ldots, Y_k \ i.i.d. \ \text{with} \ Y_\ell \sim \text{Ber}(a_0), \ \ell = 1, \ldots, k \]

with \( a_1 \neq a_0 \) in \( (0, 1) \). What are the likelihood ratio tests for this binary hypothesis testing problem?

9. In the context of Problem 8 can the Central Limit Theorem be used to compute \( P_F(Lrt_\eta) \) and \( P_M(Lrt_\eta) \) with \( \eta > 0 \) when \( k \) is large? Explain!