Fact: For any Borel mapping \( u : \mathbb{R} \to \mathbb{R} \) which is integrable, i.e.,
\[
\int_{\mathbb{R}} |u(y)| dy < \infty
\]
we necessarily have
\[
\lim_{y \to \pm \infty} |u(y)| = 0.
\]

1.a. The point of this question was to extract the conditions on the probability density function \( h : \mathbb{R} \to \mathbb{R}_+ \) which ensure that the conditions (CR1)-(CR5) hold for the family \( \{f_\theta, \ \theta \in \mathbb{R}\} \). They are

(i) The support of \( h : \mathbb{R} \to \mathbb{R}_+ \) is the entirely line \( \mathbb{R} \), namely
\[
h(y) > 0, \quad y \in \mathbb{R}. \tag{1.1}
\]

(ii) The probability density function \( h : \mathbb{R} \to \mathbb{R}_+ \) is differentiable everywhere on \( \mathbb{R} \).

(iii) The square-integrability condition
\[
\int_{\mathbb{R}} \left| \frac{h'(y)}{h(y)} \right|^2 h(y) dy < \infty \tag{1.2}
\]
holds.

(iv) The derivative \( h' : \mathbb{R} \to \mathbb{R} \) is integrable in that
\[
\int_{\mathbb{R}} |h'(y)| dy < \infty. \tag{1.3}
\]
(CR1) This condition is automatically satisfied here since $\Theta = (0, \infty)$ is an open set in $\mathbb{R}$.

(CR2) For each $\theta$ in $\mathbb{R}$, $F_\theta$ is absolutely continuous with respect to Lebesgue measure with probability density function $f_\theta : \mathbb{R} \to \mathbb{R}_+$ given by

$$f_\theta(y) = h(y - \theta), \quad y \in \mathbb{R}$$

and (CR2a) holds. The support $S(\theta)$ of the probability density function $f_\theta : \mathbb{R} \to \mathbb{R}_+$ is given by

$$S(\theta) = \{y \in \mathbb{R} : f_\theta(y) > 0\} = \{y \in \mathbb{R} : h(y - \theta) > 0\}$$

so that

$$S(\theta) = \theta + S(0), \quad \theta \in \mathbb{R}$$

It follows that we must have $S(\theta) = \mathbb{R}$ for each $\theta$ in $\mathbb{R}$ and (CR2b) holds with $S = \mathbb{R}$. This obviously requires that (1.1) holds.

(CR3) Assuming the existence of needed derivatives we get

$$\frac{\partial}{\partial \theta} f_\theta(y) = -h'(y - \theta), \quad \theta \in \mathbb{R}, \quad y \in \mathbb{R}$$

where $h' : \mathbb{R} \to \mathbb{R}$ is the derivative of $h$. Thus, (CR3) requires that the probability density function $h : \mathbb{R} \to \mathbb{R}_+$ be differentiable everywhere on $\mathbb{R}$.

(CR4) This integrability condition reads

$$\mathbb{E}_\theta \left[ \left| \frac{\partial}{\partial \theta} \log f_\theta(Y) \right|^2 \right] = \int_{\mathbb{R}} \left| \frac{h'(y - \theta)}{h(y - \theta)} \right|^2 h(y - \theta) dy < \infty, \quad \theta \in \mathbb{R},$$

and reduces to the single integrability condition (1.2).

(CR5) This regularity condition amounts to

$$0 = \int_{\mathbb{R}} \frac{\partial}{\partial \theta} f_\theta(y) dy, \quad \theta \in \mathbb{R}$$

i.e.,

$$0 = \int_{\mathbb{R}} h'(y - \theta) dy, \quad \theta \in \mathbb{R}$$

Thus, by a simple of variable we see that (CR5) holds provided the integrability condition (1.3) holds for the derivative $h' : \mathbb{R} \to \mathbb{R}$ with

$$0 = \int_{\mathbb{R}} h'(y) dy.$$

Note that

$$\int_{\mathbb{R}} h'(y) dy = \lim_{A,B \to \infty} \int_{-A}^{B} h'(y) dy = \lim_{A,B \to \infty} (h(B) - h(-A)) = 0$$
and this always holds by virtue of the assumed integrability (1.3).

1.b. It is plain that the Fisher information matrix is given

\[
M(\theta) = \mathbb{E}_\theta \left( \left[ \frac{\partial}{\partial \theta} \log f_\theta(Y) \right]^2 \right)
\]

\[
= \int_\mathbb{R} \left( \frac{h'(y - \theta)}{h(y - \theta)} \right)^2 h(y - \theta) dy
\]

\[
= \int_\mathbb{R} \left( \frac{h'(z)}{h(z)} \right)^2 h(z) dz, \quad \theta \in \mathbb{R}.
\]  

(1.4)

This quantity does not depend on \( \theta \).

1.c. Fix \( \theta \) in \( \mathbb{R} \). Under the integrability condition on \( h : \mathbb{R} \to \mathbb{R}_+ \), namely

\[
\int_\mathbb{R} |z|h(z)dz < \infty
\]  

(1.5)

the integral

\[
\mu_h = \int_\mathbb{R} z h(z) dz
\]

is well defined and finite – In fact \( \mu_h \) is simply the first moment under the probability density function \( h : \mathbb{R} \to \mathbb{R}_+ \). Hence, the estimator \( g : \mathbb{R} \to \mathbb{R} \) is a finite mean estimator since

\[
\mathbb{E}_\theta [g(Y)] = \int_\mathbb{R} g(y)f_\theta(y)dy
\]

\[
= \int_\mathbb{R} (ay + b)h(y - \theta)dy
\]

\[
= \int_\mathbb{R} (az + (a\theta + b))h(z)dz \quad [z = y - \theta]
\]

\[
= a\mu_h + (a\theta + b)
\]  

(1.6)

It follows that

\[
\frac{d}{d\theta} \mathbb{E}_\theta [g(Y)] = a.
\]

On the other hand,

\[
\int_\mathbb{R} g(y) \frac{\partial}{\partial \theta} f_\theta(y)dy = a \int_\mathbb{R} y \frac{\partial}{\partial \theta} f_\theta(y)dy + b \int_\mathbb{R} \frac{\partial}{\partial \theta} f_\theta(y)dy
\]

\[
= a \int_\mathbb{R} y \frac{\partial}{\partial \theta} f_\theta(y)dy
\]  

(1.7)

since

\[
\int_\mathbb{R} \frac{\partial}{\partial \theta} f_\theta(y)dy = 0
\]
by the regularity condition (CR5). Next, we see that
\[
\int_{\mathbb{R}} y \frac{\partial}{\partial \theta} f_\theta(y) dy = \int_{\mathbb{R}} y (-h'(y - \theta)) dy \\
= - \int_{\mathbb{R}} (z + \theta) h'(z) dz
\]
under the condition \( \int_{\mathbb{R}} |z||h'(z)| dz < \infty. \)

Here as well we have
\[
\int_{\mathbb{R}} h'(z) dz = \lim_{A,B \to \infty} \int_{-A}^{B} h'(z) dz = \lim_{A,B \to \infty} (h(B) - h(-A)) = 0
\]
as before. On the other hand, integration by parts gives
\[
\int_{\mathbb{R}} zh'(z) dz = \lim_{A,B \to \infty} \int_{-A}^{B} zh'(z) dz \\
= \lim_{A,B \to \infty} \left( [zh(z)]_{-A}^{B} - \int_{A}^{B} h(z) dz \right) \\
= \lim_{A,B \to \infty} (Bh(B) + Ah(-A) - (H(B) - H(-A))) = -1
\]
because \( \lim_{z \to \infty} |z||h(z)| = 0 \) as a result of the integrability condition (1.5); see Fact. Therefore,
\[
\int_{\mathbb{R}} g(y) \frac{\partial}{\partial \theta} f_\theta(y) dy = a,
\]
and the affine estimator is indeed regular under the integrability condition (1.5).

2.

2.a. For each \( \theta > 0, \)
\[
1 = \int_{\mathbb{R}} K(\theta)e^{-\frac{y^2}{\theta^2}} dy \\
= \int_{\mathbb{R}} \theta K(\theta)e^{-\frac{z^2}{\theta}} \theta^{-1} dy \\
= \int_{\mathbb{R}} \theta K(\theta)e^{-\frac{z^2}{\theta}} dz \quad \left[ z = \frac{y}{\theta} \right] \\
= \frac{\theta K(\theta)}{K(1)} \int_{\mathbb{R}} K(1)e^{-z^2} dz.
\]
It follows that
\[
1 = \frac{\theta K(\theta)}{K(1)}.
\]
whence
\[ K(\theta) = \frac{K(1)}{\theta}. \]

2.b. Fix \( n = 1, 2, \ldots \) and \( \theta > 0 \). For arbitrary \( y_1, \ldots, y_n \) in \( \mathbb{R} \), we have
\[
\begin{align*}
 f_{\theta}(y_1, \ldots, y_n) &= \prod_{i=1}^{n} K(\theta)e^{-\frac{y_i^4}{4\theta}} \\
 &= K(\theta)^n e^{-\frac{1}{\theta^4} \sum_{i=1}^{n} y_i^4} \\
 &= K(1)^n \theta^{-n} e^{-\frac{1}{\theta^4} \sum_{i=1}^{n} y_i^4} \\
 &= K(1)^n e^{-n \log \theta - \frac{1}{\theta^4} \sum_{i=1}^{n} y_i^4}.
\end{align*}
\]

Thus,
\[
\frac{\partial}{\partial \theta} \log f_{\theta}(y_1, \ldots, y_n) = -\frac{n}{\theta} + 4 \sum_{i=1}^{n} y_i^4.
\]

The ML equation
\[
\frac{\partial}{\partial \theta} \log f_{\theta}(y_1, \ldots, y_n) = 0, \quad \theta > 0
\]

has a unique solution
\[
g_{\text{ML}}(y_1, \ldots, y_n) = \sqrt[4]{4 \sum_{i=1}^{n} y_i^4}.
\]

2.c. By the SLLNs for the rvs \( \{Y_i^4, i = 1, 2, \ldots\} \) (under \( \mathbb{P}_\theta \)) it holds that
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} Y_i^4 = \mathbb{E}_\theta[Y^4] \quad \mathbb{P}_\theta - \text{a.s.},
\]

whence
\[
\lim_{n \to \infty} g_{\text{ML}}(Y_1, \ldots, Y_n) = \sqrt[4]{4 \mathbb{E}_\theta[Y^4]} \quad \mathbb{P}_\theta - \text{a.s.}
\]

and it remains to evaluate \( \mathbb{E}_\theta[Y^4] \).

We have
\[
\mathbb{E}_\theta[Y^4] = \int_{\mathbb{R}} K(\theta) y^4 e^{-\frac{y^4}{4\theta}} dy
\]
\[
= K(\theta) \theta^5 \int_{\mathbb{R}} \frac{y^4}{\theta^4} e^{-\frac{y^4}{4\theta}} \theta^{-1} dy
\]
\[
= K(\theta) \theta^5 \int_{\mathbb{R}} z^4 e^{-z^4} \, dz \quad [z = \frac{y}{\theta}] \quad (1.14)
\]
\[
= -\frac{K(\theta)}{4} \theta^5 \int_{\mathbb{R}} z \cdot (-4z^3 e^{-z^4}) \, dz
\]
\[
= -\frac{K(\theta)}{4} \theta^5 \int_{\mathbb{R}} z \cdot (e^{-z^4})' \, dz \quad (1.15)
\]
with
\[
\int_{\mathbb{R}} z \cdot (e^{-z^4})' \, dz = \left[ z e^{-z^4} \right]_{-\infty}^{\infty} - \int_{\mathbb{R}} e^{-z^4} \, dz = -\int_{\mathbb{R}} e^{-z^4} \, dz = -K(1)^{-1}
\]
by integration by parts. Collecting we conclude that
\[
\mathbb{E}_\theta [Y^4] = -\frac{K(\theta)}{4} \theta^4 (-K(1)^{-1}) = \frac{K(\theta)}{4} \theta^4 K(1)^{-1} = \frac{\theta^4}{4} \tag{1.16}
\]
as we use the fact \(K(\theta) = K(1)\theta^{-1}\) established in Part a, so that
\[
\sqrt[4]{4\mathbb{E}_\theta [Y^4]} = \theta.
\]
The ML estimator is therefore strongly consistent.

3.

3.a. Here, for each \(\theta > 0\),
\[
f^{(3)}_\theta(y_1, y_2, y_3) = \frac{1}{8} e^{-\sum_{i=1}^3 |y_i - \theta|}, \quad y_1, y_2, y_3 \in \mathbb{R}
\]
To find the ML estimator \(g_{\text{ML}} : \mathbb{R}^3 \to (0, \infty)\) we proceed as follows: With observations \(y_1, y_2, y_3\) given, we seek to find \(g_{\text{ML}}(y_1, y_2, y_3) > 0\) such that
\[
\sum_{i=1}^3 |y_i - g_{\text{ML}}(y_1, y_2, y_3)| \leq \sum_{i=1}^3 |y_i - \theta|, \quad \theta > 0.
\]
Note that here \(\theta > 0\), and not \(\theta\) unconstrained in \(\mathbb{R}\)! For this unconstrained version of the problem we need to find \(g_{\text{ML}}(y_1, y_2, y_3)\) in \(\mathbb{R}\) such that
\[
\sum_{i=1}^3 |y_i - g_{\text{ML}}(y_1, y_2, y_3)| \leq \sum_{i=1}^3 |y_i - \theta|, \quad \theta \in \mathbb{R}.
\]
Its solution is well known and can be described as follows: Given the values \(y_1, y_2, y_3\) in \(\mathbb{R}\), write \(y_{(1)}, y_{(2)}, y_{(3)}\) for these values ordered in increasing values (with a lexicographic tiebreaker), i.e., \(\{y_{(1)}, y_{(2)}, y_{(3)}\} = \{y_1, y_2, y_3\}\) with
\[
y_{(1)} \leq y_{(2)} \leq y_{(3)}.
\]
Then, with this notation we have
\[ g_{ML}(y_1, y_2, y_3) = y(2). \]

It can be interpreted as the median for the uniform distribution on \{y_1, y_2, y_3\}!

To see why this is indeed true, observe that (i) \( g_{ML}(y_1, y_2, y_3) \) must necessarily lie in the interval \( y(1), y(3) \) - the metric of interest can always be decreased otherwise by moving towards the boundary points \( y(1) \) or \( y(3) \); and (ii) With \( a < b \), we have
\[ |a - \theta| + |b - \theta| = a - b, \quad \theta \in [a, b], \]
a fact which argues for the solution to necessarily be at \( y(2) \).

It is easy to see by a symmetry argument that
\[ E_\theta [Y(2)] = \theta, \quad \theta \in \mathbb{R}. \]

Just take expectations in the identity
\[ \sum_{i=1}^{3} Y_i = \sum_{i=1}^{3} Y(i) \]
and use the fact that for each \( \theta \) in \( \mathbb{R} \), we have
\[ (Y(1) - \theta) =_{st} (Y(3) - \theta) \]
under \( P_\theta \).

However, here we need to solve the constrained problem: Find \( g_{ML}(y_1, y_2, y_3) > 0 \) such that
\[ \sum_{i=1}^{3} |y_i - g_{ML}(y_1, y_2, y_3)| \leq \sum_{i=1}^{3} |y_i - \theta|, \quad \theta > 0. \]

Four cases need to be considered:
(i) If \( y(3) \leq 0 \), then the ML estimate \( g_{ML}(y_1, y_2, y_3) \) does not exist (at least in the strict sense as an element of \( (0, \infty) \)). However one may decide to allow the search to be carried out over the larger set \( \mathbb{R}_+ \) (thereby including the boundary point \( \theta = 0 \)), in which case \( g_{ML}(y_1, y_2, y_3) = 0 \).
(ii) If \( y(2) \leq 0 < y(3) \), then \( g_{ML}(y_1, y_2, y_3) = 0 \) (in the extended formulation, otherwise it does not exist).
(iii) If \( y(1) \leq 0 < y(2) \), then \( g_{ML}(y_1, y_2, y_3) = y(2) \) (in the original formulation).
(iv) If \( 0 < y(1) \), then \( g_{ML}(y_1, y_2, y_3) = y(2) \) (in the original formulation).

3.b. The ML estimator cannot be an MVUE estimator since obviously
\[ E_\theta [g_{ML}(Y_1, Y_2, Y_3)] \neq \theta > 0 \]
by remarks above as we note that
\[ Y(2) \leq g_{ML}(Y_1, Y_2, Y_3). \]
3.c. The family \( \{ F_\theta, \theta > 0 \} \) is not an exponential family as can be checked by direct inspection (in spite of its “exponential nature”).

4.  

4.a. Fix \( t \geq 0 \) and \( y = 0, 1, \ldots \). The posterior distribution of \( \vartheta \) given \( Y = y \) is easily computed as

\[
f_{\vartheta|Y}(t|y) = \frac{\nu^y e^{-t} g(t)}{\mathbb{P}[Y = y]}, \quad t \geq 0, \quad y = 0, 1, \ldots \tag{1.18}
\]

with

\[
\mathbb{P}[Y = y] = \int_0^\infty \frac{\tau^y}{y!} e^{-\tau} g(\tau) d\tau, \quad y = 0, 1, \ldots
\]

Therefore, for each \( y = 0, 1, \ldots \), we get

\[
\mathbb{E}[\vartheta|Y = y] = \int_0^\infty t f_{\vartheta|Y}(t|y) dt
\]

\[
= \frac{\int_0^\infty \nu^y e^{-t} g(t) dt}{\int_0^\infty \nu^y e^{-t} g(t) dt} = \frac{W(y+1)}{W(y)} \tag{1.19}
\]

with

\[
W(y) = \mathbb{E}[\vartheta e^{-\vartheta}] = \int_0^\infty t^y e^{-t} g(t) dt, \quad y = 0, 1, \ldots
\]

4.b. Alternatively,

\[
\mathbb{E}[\vartheta|Y = y] = (y + 1) \frac{\mathbb{P}[Y = y+1]}{\mathbb{P}[Y = y]}, \quad y = 0, 1, \ldots
\]

4.c. It is well known that

\[
\hat{\mathbb{E}}[\vartheta|Y = y] = \mu_\vartheta + \frac{\Sigma_{\vartheta Y}}{\Sigma_Y} (y - \mu_Y), \quad y \in \mathbb{R}.
\]

Note that

\[
\mathbb{E}[Y^p|\vartheta] = \begin{cases} 
\vartheta & \text{if } p = 1 \\
\vartheta^2 + \vartheta & \text{if } p = 2
\end{cases}
\]

by standard properties of the Poisson distribution.

By standard preconditioning arguments it follows that

\[
\mu_Y = \mathbb{E}[Y] = \mathbb{E} [\mathbb{E}[Y|\vartheta]] = \mathbb{E}[\vartheta], \tag{1.20}
\]

and

\[
\Sigma_Y = \text{Var}[Y] = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2
\]

\[
= \mathbb{E}[\mathbb{E}[Y^2|\vartheta]] - (\mathbb{E}[\vartheta])^2
\]

\[
= \mathbb{E}[\vartheta^2 + \vartheta] - (\mathbb{E}[\vartheta])^2
\]

\[
= \text{Var}[\vartheta] + \mathbb{E}[\vartheta]. \tag{1.21}
\]
In a similar vein, we have

\[
\Sigma_{\theta Y} = \text{Cov}[\theta, Y] \\
= \mathbb{E}[\theta Y] - \mathbb{E}[\theta] \mathbb{E}[Y] \\
= \mathbb{E}[\theta \mathbb{E}[Y|\theta]] - (\mathbb{E}[\theta])^2 \\
= \mathbb{E}[\theta^2] - (\mathbb{E}[\theta])^2 \\
= \text{Var}[\theta].
\] (1.22)

Collecting,

\[
\hat{\mathbb{E}}[\theta|Y = y] = \mathbb{E}[\theta] + \frac{\text{Var}[\theta]}{\text{Var}[\theta] + \mathbb{E}[\theta]} (y - \mathbb{E}[\theta]) \\
= \frac{\text{Var}[\theta]}{\text{Var}[\theta] + \mathbb{E}[\theta]} y + \frac{(\mathbb{E}[\theta])^2}{\text{Var}[\theta] + \mathbb{E}[\theta]}, \quad y \in \mathbb{R}.
\] (1.23)

5. 

5.a. Here \( \Theta = \{0, 1, \ldots, M - 1\} \). With \( \theta = 0, 1, \ldots, M - 1, \) we have

\[
f_{\theta}^{(n)}(y_1, \ldots, y_n) = \prod_{i=1}^{n} f_\theta(y_i) \\
= \exp \sum_{i=1}^{n} \log f_\theta(y_i) \\
= \exp \sum_{k=0}^{M-1} 1[\theta = k] (\sum_{i=1}^{n} \log f_k(y_i)) \\
= C_n(\theta) q_n(y_1, \ldots, y_n) e^{Q_n(\theta)' K_n(y_1, \ldots, y_n)}
\] (1.24)

where for each \( \theta \) in \( \Theta \), we have set

\[ C_n(\theta) = 1 \quad \text{and} \quad Q_n(\theta) = (1[\theta = 0], \ldots, 1[\theta = M - 1])' \]

while with \( y_1, \ldots, y_n \) in \( \mathbb{R} \),

\[ q_n(y_1, \ldots, y_n) = 1 \]

and

\[ K_n(y_1, \ldots, y_n) = \left( \sum_{i=1}^{n} \log f_0(y_i), \ldots, \sum_{i=1}^{n} \log f_{M-1}(y_i) \right)' . \]

It is plain from (1.24) that the family \( \{F_m^{(n)}, m = 0, \ldots, M - 1\} \) is an exponential family.

5.b. There are \( M \) natural sufficient statistics \( T_0, \ldots, T_{M-1} : \mathbb{R}^n \to \mathbb{R} \) given by

\[ T_m(y_1, \ldots, y_n) = \sum_{i=1}^{n} \log f_m(y_i), \quad y_1, \ldots, y_n \in \mathbb{R} \]

\[ m = 0, \ldots, M - 1. \]

This set of sufficient statistics are marginally interesting as they are equivalent to the statistics \( f_0^{(n)}, \ldots, f_{M-1}^{(n)} : \mathbb{R}^n \to \mathbb{R}_+ \); however they do reduce dimensionality from \( n \) (number of observations) to \( M \) (number of hypotheses)!
6. **6.a.** With $\eta > 0$, write $\eta = e^{-\tau}$ for some $\tau$ in $\mathbb{R}$. The test $d_\eta : \mathbb{R} \to \{0, 1\}$ is then given by

$$d_\eta(y) = 0 \iff f_\lambda(y) < \eta f_0(y).$$

This reduces to

$$d_\eta(y) = 0 \iff \tau + |y| < |y - \lambda|$$

so that

$$C(d_\eta) = \{ y \in \mathbb{R} : \tau + |y| < |y - \lambda| \}$$

with $\eta = e^{-\tau}$.

Three separate cases need to be considered. An easy geometric argument shows the following:

(i) If $\tau < -\lambda$, then $C(d_\eta) = \mathbb{R}$.

(ii) If $-\lambda \leq \tau < \lambda$, then

$$C(d_\eta) = \left( -\infty, \frac{\lambda - \tau}{2} \right).$$

(iii) If $\lambda \leq \tau$, then $C(d_\eta)$ is empty.

**6.b.** Using the results of Part a, we conclude the following: If $\tau < -\lambda$, then

$$P_F(d_\eta) = \mathbb{P} [d_\eta(Y) = 1 | H = 0] = 0,$$

and

$$P_D(d_\eta) = \mathbb{P} [d_\eta(Y) = 1 | H = 1] = 0.$$

If $\lambda \leq \tau$, then

$$P_F(d_\eta) = \mathbb{P} [d_\eta(Y) = 1 | H = 0] = 1,$$

and

$$P_D(d_\eta) = \mathbb{P} [d_\eta(Y) = 1 | H = 1] = 1.$$

If $-\lambda \leq \tau < \lambda$, then

$$P_F(d_\eta) = \mathbb{P} [d_\eta(Y) = 1 | H = 0] = \mathbb{P} \left[ Y \geq \frac{\lambda - \tau}{2} | H = 0 \right] = \int_{\frac{\lambda - \tau}{2}}^{\infty} f_0(y) dy = \int_{\frac{\lambda - \tau}{2}}^{\infty} \frac{1}{2} e^{-|y|} dy = \frac{1}{2} e^{-\frac{\lambda - \tau}{\tau}}$$

(1.25)
and

\[ P_D(d_\eta) = \mathbb{P}[d_\eta(Y) = 1|H = 1] = \mathbb{P}[Y \geq \frac{\lambda - \tau}{2}|H = 1] = \int_{\frac{\lambda - \tau}{2}}^{\infty} f_\lambda(y) dy = \int_{\frac{\lambda - \tau}{2}}^{\infty} \frac{1}{2} e^{-|y-\lambda|} dy. \quad (1.26) \]

In computing this last integral, we note that

\[ 0 < \frac{\lambda - \tau}{2} \leq \lambda \text{ if } -\lambda \leq \tau < \lambda. \]

Therefore,

\[ \int_{\frac{\lambda - \tau}{2}}^{\infty} e^{-|y-\lambda|} dy = \int_{\frac{\lambda - \tau}{2}}^{\lambda} e^{-|y-\lambda|} dy + \int_{\lambda}^{\infty} e^{-|y-\lambda|} dy = \int_{\frac{\lambda - \tau}{2}}^{\lambda} e^{-(\lambda-y)} dy + \int_{\lambda}^{\infty} e^{-(y-\lambda)} dy = e^{-\lambda} \left( e^{\lambda} - e^{\frac{\lambda - \tau}{2}} \right) + 1 = 2 - e^{-\frac{\lambda + \tau}{2}}, \quad (1.27) \]

whence

\[ P_D(d_\eta) = 1 - \frac{1}{2} e^{-\frac{\lambda + \tau}{2}}. \]

6.c. To compute the ROC curve, first we note that

\[ \{P_F(d_\eta), \tau < -\lambda\} = \{P_D(d_\eta), \tau < -\lambda\} = \{0\} \]

and

\[ \{P_F(d_\eta), \lambda \leq \tau\} = \{P_D(d_\eta), \lambda \leq \tau\} = \{1\}, \]

while

\[ \{P_F(d_\eta), -\lambda \leq \tau < \lambda\} = \left\{ \frac{1}{2} e^{-\frac{\lambda + \tau}{2}}, -\lambda \leq \tau < \lambda \right\} = \left[ \frac{e^{-\lambda}}{2}, \frac{1}{2} \right] \]

and

\[ \{P_D(d_\eta), -\lambda \leq \tau < \lambda\} = \left\{ 1 - \frac{1}{2} e^{-\frac{\lambda + \tau}{2}}, -\lambda \leq \tau < \lambda \right\} = \left( \frac{1}{2}, 1 - \frac{1}{2} e^{-\lambda} \right). \]

With \( P_F \) in \( \left[ \frac{e^{-\lambda}}{2}, \frac{1}{2} \right] \) we solve the equation

\[ \frac{1}{2} e^{-\frac{\lambda + \tau}{2}} = P_F, \quad -\lambda \leq \tau < \lambda \]
It has a unique solution $\tau(P_F)$ given by

$$\tau(P_F) = \lambda + \log(4P_F^2).$$

Note that

$$\tau(P_F) \in [-\lambda, \lambda)$$

by direct inspection (as expected). Therefore, with $\eta = e^{-\tau(P_F)}$, we find

$$P_D(d_\eta) = 1 - \frac{1}{2}e^{-\frac{\lambda + \tau(P_F)}{2}}$$

$$= 1 - \frac{1}{2}e^{-\frac{\lambda + \lambda + \log(4P_F^2)}{2}}$$

$$= 1 - \frac{1}{2}e^{-\lambda - \log(2P_F)}$$

$$= 1 - \frac{e^{-\lambda}}{4P_F}.$$  \hfill (1.28)

From the discussion it follows that the mapping $\Gamma$ is not defined on the entire interval $[0, 1]$. In fact, we have

$$\Gamma(P_F) = \begin{cases} 
0 & \text{if } P_F = 0 \\
1 - \frac{e^{-\lambda}}{4P_F} & \text{if } P_F \in \left(\frac{e^{-\lambda}}{2}, \frac{1}{2}\right) \\
1 & \text{if } P_F = 1.
\end{cases}$$

As always the ROC curve can be completed through linear interpolation achieved through randomization.