1. For each \( h = 0, 1 \), the probability distribution \( F_h \) has probability density function \( f_h : \mathbb{R} \to \mathbb{R}_+ \) given by

\[
f_h(y) = \begin{cases} 
0 & \text{if } y < 0 \\
\alpha_h e^{-\alpha_h y} & \text{if } y \geq 0.
\end{cases}
\]

Therefore,

\[
d_\eta(y) = 0 \quad \text{iff} \quad f_1(y) < \eta f_0(y) \\
\text{iff} \quad \alpha_1 e^{-\alpha_1 y} < \eta \alpha_0 e^{-\alpha_0 y}, \ y \geq 0 \\
\text{iff} \quad e^{-(\alpha_1 - \alpha_0)y} < \frac{\alpha_0}{\alpha_1}, \ y \geq 0 \\
\text{iff} \quad \log \left( \frac{\alpha_1}{\eta \alpha_0} \right) < (\alpha_1 - \alpha_0)y, \ y \geq 0 \\
\text{iff} \quad \frac{1}{\alpha_1 - \alpha_0} \log \left( \frac{\alpha_1}{\eta \alpha_0} \right) < y, \ y \geq 0.
\]

(1.1)

It is plain that

\[
C(d_\eta) = \left\{ y \geq 0 : \frac{1}{\alpha_1 - \alpha_0} \log \left( \frac{\alpha_1}{\eta \alpha_0} \right) < y \right\}.
\]

(1.2)

1.b. Obviously,

\[
P_F(d_\eta) = \mathbb{P}[d_\eta(Y) = 1|H = 0] \\
= 1 - \mathbb{P}[d_\eta(Y) = 0|H = 0] \\
= 1 - \mathbb{P} \left[ \frac{1}{\alpha_1 - \alpha_0} \log \left( \frac{\alpha_1}{\eta \alpha_0} \right) < Y|H = 0 \right] \\
= 1 - e^{-\frac{\alpha_0}{\alpha_1 - \alpha_0} \left( \log \left( \frac{\alpha_1}{\eta \alpha_0} \right) \right)^+}.
\]

(1.3)
In a similar way, we get
\[
P_D(d_\eta) = \mathbb{P}[d_\eta(Y) = 1|H = 1] = 1 - \mathbb{P}[d_\eta(Y) = 0|H = 1] = 1 - \mathbb{P}\left[\frac{1}{\alpha_1 - \alpha_0} \log \left( \frac{\alpha_1}{\eta\alpha_0} \right) < Y|H = 1 \right] = 1 - e^{-\frac{\alpha_0}{\alpha_1 - \alpha_0} \left( \log \left( \frac{\alpha_1}{\eta\alpha_0} \right) \right)^+}.
\]

1.c. With the notation introduced in the Lecture Notes we have
\[
V(p) = J_p(d^*(p)) = J_p(d_{\eta(p)}), \quad p \in [0, 1]
\]
where
\[
\eta(p) = \frac{\Gamma_0(1 - p)}{\Gamma_1 p} = \frac{1 - p}{p}
\]
since here \( \Gamma_0 = \Gamma_1 = 1 \). It is now straightforward to see that
\[
V(p) = p\mathbb{P}\left[d_{\eta(p)}(Y) = 0|H = 1 \right] + (1 - p)\mathbb{P}\left[d_{\eta(p)}(Y) = 1|H = 0 \right]
= p \left(1 - P_D(d_{\eta(p)}) \right) + (1 - p)P_F(d_{\eta(p)})
\]
\[
= pe^{-\frac{\alpha_0}{\alpha_1 - \alpha_0} \left( \log \left( \frac{\alpha_1}{\eta\alpha_0} \right) \right)^+} + (1 - p) \left(1 - e^{-\frac{\alpha_0}{\alpha_1 - \alpha_0} \left( \log \left( \frac{\alpha_1}{\eta\alpha_0} \right) \right)^+} \right) = \begin{cases} 
pe^{-\frac{\alpha_0}{\alpha_1 - \alpha_0} \left( \log \left( \frac{\alpha_1}{\eta\alpha_0} \right) \right)^+} + (1 - p) \left(1 - e^{-\frac{\alpha_0}{\alpha_1 - \alpha_0} \left( \log \left( \frac{\alpha_1}{\eta\alpha_0} \right) \right)^+} \right) & \text{if } \eta(p)\alpha_0 < \alpha_1 \\
\frac{\eta(p)\alpha_0}{\alpha_1} + (1 - p) \left(1 - \left( \frac{\eta(p)\alpha_0}{\alpha_1} \right)^{\frac{\alpha_0}{\alpha_1 - \alpha_0}} \right) & \text{if } \eta(p)\alpha_0 < \alpha_1 \\
p & \text{if } \alpha_1 \leq \eta(p)\alpha_0 \\
p \left(1 - \frac{\eta\alpha_0}{\alpha_1} \right)^{\frac{\alpha_0}{\alpha_1 - \alpha_0}} + (1 - p) \left(1 - \left(1 - \frac{\eta\alpha_0}{\alpha_1} \right)^{\frac{\alpha_0}{\alpha_1 - \alpha_0}} \right) & \text{if } \frac{\eta\alpha_0}{\alpha_1 + \alpha_0} < p \leq 1 \\
p & \text{if } 0 \leq p \leq \frac{\eta\alpha_0}{\alpha_0 + \alpha_1}.
\end{cases}
\]

1.d. First note that
\[
P_F(d_{\eta}) = 1 - e^{-\frac{\alpha_0}{\alpha_1 - \alpha_0} \left( \log \left( \frac{\alpha_1}{\eta\alpha_0} \right) \right)^+} = \begin{cases} 
1 - e^{-\frac{\alpha_0}{\alpha_1 - \alpha_0} \left( \log \left( \frac{\alpha_1}{\eta\alpha_0} \right) \right)^+} & \text{if } \eta < \frac{\alpha_1}{\alpha_0} \\
0 & \text{if } \frac{\alpha_1}{\alpha_0} \leq \eta \\
1 - \left( \frac{\eta\alpha_0}{\alpha_1} \right)^{\frac{\alpha_0}{\alpha_1 - \alpha_0}} & \text{if } \eta < \frac{\alpha_1}{\alpha_0} \\
0 & \text{if } \frac{\alpha_1}{\alpha_0} \leq \eta
\end{cases}
\]
Fix $P_F$ in $(0, 1]$ and solve the equation

$$P_F(d_\eta) = P_F, \quad \eta \geq 0.$$  

In view of the previous calculations, this amounts to solving

$$(\frac{\eta \alpha_0}{\alpha_1})^{\frac{\alpha_0}{\alpha_1 - \alpha_0}} = 1 - P_F, \quad 0 \leq \eta < \frac{\alpha_1}{\alpha_0}.$$  

This has a unique solution $\eta(P_F)$ given by

$$\eta(P_F) = \frac{\alpha_1}{\alpha_0} (1 - P_F)^{\frac{\alpha_1 - \alpha_0}{\alpha_0}}.$$  

The corresponding point $P_D$ on the ROC curve is therefore given by $P_D(d_\eta(P_F))$ evaluated as

$$P_D(d_\eta(P_F)) = 1 - e^{-\frac{\alpha_1}{\alpha_1 - \alpha_0} \left( \log \left( \frac{\alpha_1}{\eta(P_F) \alpha_0} \right) \right)}.$$  

But

$$\frac{\alpha_1}{\eta(P_F) \alpha_0} = \frac{\alpha_1}{\alpha_0} \cdot \frac{\alpha_1}{1 - P_F} \frac{\alpha_1 - \alpha_0}{\alpha_0} = (1 - P_F)^{\frac{\alpha_1 - \alpha_0}{\alpha_0}} > 1$$  

and

$$\log \left( \frac{\alpha_1}{\eta(P_F) \alpha_0} \right) = \log (1 - P_F)^{-\frac{\alpha_1 - \alpha_0}{\alpha_0}} = -\frac{\alpha_1 - \alpha_0}{\alpha_0} \log (1 - P_F) > 0.$$  

Therefore,

$$P_D(d_\eta(P_F)) = 1 - e^{-\frac{\alpha_1}{\alpha_1 - \alpha_0} \left( \log \left( \frac{\alpha_1}{\eta(P_F) \alpha_0} \right) \right)}$$

$$= 1 - e^{-\frac{\alpha_1}{\alpha_1 - \alpha_0} \left( -\frac{\alpha_1 - \alpha_0}{\alpha_0} \log (1 - P_F) \right)}$$

$$= 1 - e^{\frac{\alpha_1}{\alpha_0} \log (1 - P_F)}$$

$$= 1 - (1 - P_F)^{\frac{\alpha_1}{\alpha_0}}.$$  

(1.4)

We conclude that $\Gamma : [0, 1] \to [0, 1]$ is given by

$$\Gamma(P_F) = 1 - (1 - P_F)^{\frac{\alpha_1}{\alpha_0}}, \quad P_F \in [0, 1].$$  

2.

Recall that a rv $Y$ is said to be Rayleigh distributed with parameter $\theta > 0$ if its probability distribution $F_{\theta}$ admits a probability density function $f_{\theta} : \mathbb{R} \to \mathbb{R}_+$ given by

$$f_{\theta}(y) = \begin{cases} 
0 & \text{if } y < 0 \\
\frac{y}{\theta^2} e^{-\frac{y^2}{2\theta^2}} & \text{if } y \geq 0.
\end{cases}$$
It is crucial to observe that
\[
F_\theta(y) = \int_{-\infty}^{y} f_\theta(x) \, dx = 1 - e^{-\frac{(y^+)^2}{2\theta^2}}, \quad y \in \mathbb{R}.
\] (1.6)

In particular, for each \( t \) in \( \mathbb{R} \), we have
\[
\mathbb{P}_\theta [Y^2 > t] = e^{-\frac{t^+}{2\theta^2}}.
\]

2.a. With distinct \( \theta_0 \) and \( \theta_1 \) in \( (0, \infty) \), consider the binary hypothesis testing problem
\[
H_1 : Y \sim F_{\theta_1}, \\
H_0 : Y \sim F_{\theta_0}.
\] (1.7)

For \( \eta > 0 \), consider the corresponding test \( d_\eta : \mathbb{R} \to \{0, 1\} \). In a routine manner we find
\[
d_\eta(y) = 0 \quad \text{iff} \quad f_{\theta_1}(y) < \eta f_{\theta_0}(y)
\]
\[
\text{iff} \quad \frac{y}{\theta_1^2} e^{-\frac{y^2}{2\theta_1^2}} < \eta \frac{y}{\theta_0^2} e^{-\frac{y^2}{2\theta_0^2}}, \quad y > 0
\]
\[
\text{iff} \quad e^{-\frac{y^2}{2}} \left( \frac{1}{\theta_1^2} - \frac{1}{\theta_0^2} \right) < \eta \frac{\theta_2^2}{\theta_0^2}, \quad y > 0
\]
\[
\text{iff} \quad e^{-\frac{y^2}{2} D(\theta_1, \theta_0)} < \eta R(\theta_1, \theta_0), \quad y > 0
\]
with
\[
D(\theta_1, \theta_0) = \frac{1}{\theta_1^2} - \frac{1}{\theta_0^2} \quad \text{and} \quad R(\theta_1, \theta_0) = \frac{\theta_1^2}{\theta_0^2}.
\]

Taking logarithms on both sides, we get
\[
d_\eta(y) = 0 \quad \text{iff} \quad -2 \log (\eta R(\theta_1, \theta_0)) < D(\theta_1, \theta_0) y^2, \quad y > 0.
\] (1.8)

It follows that
\[
P_F(d_\eta) = \mathbb{P} [d_\eta(Y) = 1 | H = 0]
\]
\[
= 1 - \mathbb{P} [d_\eta(Y) = 0 | H = 0]
\]
\[
= 1 - \mathbb{P} [-2 \log (\eta R(\theta_1, \theta_0)) < D(\theta_1, \theta_0) Y^2 | H = 0].
\] (1.9)

If \( 0 < \theta_0 < \theta_1 \), then \( D(\theta_1, \theta_0) < 0 \) and \( R(\theta_1, \theta_0) > 1 \), so that
\[
P_F(d_\eta) = 1 - \mathbb{P} \left[ Y^2 < -\frac{2 \log (\eta R(\theta_1, \theta_0))}{D(\theta_1, \theta_0)} \bigg| H = 0 \right]
\]
\[
= \mathbb{P} \left[ Y^2 \geq -\frac{2 \log (\eta R(\theta_1, \theta_0))}{D(\theta_1, \theta_0)} \bigg| H = 0 \right]
\]
\[
= e^{-\frac{1}{2\theta_0^2} \left( \frac{-2 \log (\eta R(\theta_1, \theta_0))}{D(\theta_1, \theta_0)} \right)}.
\] (1.10)
With $\alpha$ in $(0,1)$, solving the equation

$$e^{-\frac{1}{2\theta_0^2}(-\frac{2\log(\eta R(\theta_1, \theta_0))}{D(\theta_1, \theta_0)})^+} = \alpha, \quad \eta > 0$$  \hfill (1.11)

requires

$$\log(\eta R(\theta_1, \theta_0)) > 0,$$

or equivalently,

$$\eta R(\theta_1, \theta_0) > 1.$$

Under that condition, we get

$$e^{-\frac{1}{2\theta_0^2}(-\frac{2\log(\eta R(\theta_1, \theta_0))}{D(\theta_1, \theta_0)})^+} = e^{\frac{\log(\eta R(\theta_1, \theta_0))}{\theta_0^2 D(\theta_1, \theta_0)}}$$ \hfill (1.12)

and the equation (1.11) becomes

$$\log(\eta R(\theta_1, \theta_0)) = \theta_0^2 D(\theta_1, \theta_0) \cdot \log \alpha.$$  

The solution $\eta(\alpha)$ satisfies

$$\eta(\alpha) R(\theta_1, \theta_0) = \alpha^{\theta_0^2 D(\theta_1, \theta_0)},$$

and is therefore given by

$$\eta(\alpha) = \frac{\alpha^{\theta_0^2 D(\theta_1, \theta_0)}}{R(\theta_1, \theta_0)}.$$  

The Neyman-Pearson test $d_{NP}(\theta_1, \theta_0; \alpha)$ of size $\alpha$ is characterized by

$$d_{NP}(\theta_1, \theta_0; \alpha)(y) = 0 \text{ iff } -2 \log(\eta(\alpha) R(\theta_1, \theta_0)) < D(\theta_1, \theta_0)y^2, \quad y > 0$$

$$\text{iff } -2\theta_0^2 D(\theta_1, \theta_0) \cdot \log \alpha < D(\theta_1, \theta_0)y^2, \quad y > 0$$

$$\text{iff } 2\theta_0^2 \cdot \log \alpha < -y^2, \quad y \geq 0$$

$$\text{iff } y^2 < -2\theta_0^2 \cdot \log \alpha, \quad y > 0.$$ \hfill (1.13)

Note that

$$C(d_{NP}(\theta_1, \theta_0; \alpha)) = \{y > 0 : y^2 < -2\theta_0^2 \cdot \log \alpha\}.$$  

On the other hand, if $0 < \theta_1 < \theta_0$, then $D(\theta_1, \theta_0) > 0$ and $R(\theta_1, \theta_0) < 1$, so that

$$P_F(d_\eta) = 1 - \mathbb{P}\left[Y^2 > \frac{-2\log(\eta R(\theta_1, \theta_0))}{D(\theta_1, \theta_0)} H = 0\right]$$

$$= 1 - e^{-\frac{1}{2\theta_0^2}(-\frac{2\log(\eta R(\theta_1, \theta_0))}{D(\theta_1, \theta_0)})^+}. \hfill (1.14)$$

With $\alpha$ in $(0,1)$, solving the equation

$$1 - e^{-\frac{1}{2\theta_0^2}(-\frac{2\log(\eta R(\theta_1, \theta_0))}{D(\theta_1, \theta_0)})^+} = \alpha, \quad \eta > 0$$ \hfill (1.15)

requires

$$\log(\eta R(\theta_1, \theta_0)) < 0,$$
or equivalently,
\[ \eta R(\theta_1, \theta_0) < 1. \]
Under that condition, we get
\[
1 - e^{-\frac{1}{2\theta_2^2} \left( -2 \log(\eta R(\theta_1, \theta_0)) \right)^+} = 1 - e^{-\frac{\log(\eta R(\theta_1, \theta_0))}{\theta_0^2 D(\theta_1, \theta_0)}}
\] (1.16)
and the equation (1.15) becomes
\[
\log(\eta R(\theta_1, \theta_0)) = \theta_2^2 D(\theta_1, \theta_0) \cdot \log(1 - \alpha).
\]
This yields
\[ \eta R(\theta_1, \theta_0) = (1 - \alpha)^{\theta_2^2 D(\theta_1, \theta_0)}, \]
and the solution \( \eta(\alpha) \) is given by
\[ \eta(\alpha) = \frac{(1 - \alpha)^{\theta_2^2 D(\theta_1, \theta_0)}}{R(\theta_1, \theta_0)}. \]
The Neyman-Pearson test \( d_{NP}(\theta_1, \theta_0; \alpha) \) of size \( \alpha \) is now characterized by
\[
d_{NP}(\theta_1, \theta_0; \alpha)(y) = 0 \quad \text{iff} \quad -2 \log(\eta(\alpha) R(\theta_1, \theta_0)) < D(\theta_1, \theta_0) y^2, \ y \geq 0
\]
\[
\text{iff} \quad -2\theta_0^2 D(\theta_1, \theta_0) \cdot \log(1 - \alpha) < D(\theta_1, \theta_0) y^2, \ y \geq 0
\]
\[
\text{iff} \quad -2\theta_0^2 \cdot \log(1 - \alpha) < y^2, \ y \geq 0. \quad (1.17)
\]
Note that
\[ C(d_{NP}(\theta_1, \theta_0; \alpha)) = \{ y \geq 0 : -\theta_0^2 \cdot \log(1 - \alpha) < y^2 \}. \]
2.b. With \( \Theta_0 = \{1\} \) and \( \theta_1 = (1, \infty) \), it is plain that there exists a UMP test of size \( \alpha \). Indeed note that
\[ C(d_{NP}(\theta_1, 1; \alpha)) = \{ y > 0 : y^2 < -2 \log \alpha \}, \quad \theta_1 > 1. \]
These tests are all Neyman-Pearson tests of size \( \alpha \) implementing the same decision regions without having to require explicit knowledge of \( \theta_1 \). All that is needed is that \( \theta_1 > 1! \)
2.c. When \( \Theta_0 = (0, 1) \) and \( \theta_1 = (1, \infty) \), there is no UMP test.

3.
3.a. In Chapter 3 we have seen that when all the hypotheses are equally likely, namely
\[ p_0 = \ldots = p_{M-1} = \frac{1}{M}, \]
the optimal test under the probability of error criterion is the Maximum Likelihood test \( d_{ML} : \mathbb{R} \to \{0, 1, \ldots, M - 1\} \) given by
\[ d_{ML}(y) = \arg \max (\ell = 0, \ldots, M - 1 : f_{\theta_\ell}(y)), \quad y \in \mathbb{R} \]
with a lexicographic tiebreaker in the event of ties. In other words,
\[
d_{\text{ML}}(y) = m \quad \text{iff} \quad f_{\theta_m}(y) = \max (f_{\theta_\ell}(y), \ell = 0, 1, \ldots, M - 1)
\]
with a lexicographic tiebreaker in the event of ties.

However, we note that
\[
\max (f_{\theta_\ell}(y), \ell = 0, 1, \ldots, M - 1) = \max (g(y - \theta_\ell), \ell = 0, 1, \ldots, M - 1) \quad \text{[By symmetry]}
\]
\[
= g(\min (|y - \theta_\ell|, \ell = 0, 1, \ldots, M - 1)) \quad \text{[By strict decreasing monotonicity on } \mathbb{R}_+].
\]

This implies that
\[
d_{\text{ML}}(y) = m \quad \text{iff} \quad |y - \theta_m| = \min (|y - \theta_\ell|, \ell = 0, 1, \ldots, M - 1)
\]
with a lexicographic tiebreaker in the event of ties. The geometric interpretation is clear: Given the observation \(y\), the test \(d_{\text{ML}}\) selects that hypothesis \(H_m\) whose parameter \(\theta_m\) is closest to \(y\) – This is sometimes known as the nearest neighbor detector.

It is plain that the nearest neighbor detector depends on \(g : \mathbb{R} \to \mathbb{R}_+\) only through conditions (i)–(iii), not on the specific form of \(g : \mathbb{R} \to \mathbb{R}_+\). For instance, the two densities
\[
g(y) = \frac{\alpha}{2} e^{-\alpha|y|}, \quad y \in \mathbb{R}
\]
and
\[
g(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}, \quad y \in \mathbb{R}
\]
will yield the same conclusion.

3.b. As in the binary case, randomization does not affect optimality in the \(M\)-ary case – This was not done in the Lecture Notes but can be easily shown by similar arguments. In particulate the ML test \(d_{\text{ML}}\) is also optimal among all admissible randomized tests. Therefore, by the optimality of \(d_{\text{ML}}\) we must have
\[
\mathbb{P}[d_{\text{ML}}(Y) \neq H] < \mathbb{P}[D_R \neq H]
\]
where \(D_R\) is the decision to flip an \(M\)-sided coin (independently of everything else) with
\[
\mathbb{P}[D_R = m] = \frac{1}{M}, \quad m = 0, \ldots, M - 1.
\]
But, assuming an arbitrary pdf $p$ for the rv $H$, we see that

$$
\mathbb{P}[D_R \neq H] = \sum_{m=0}^{M-1} \mathbb{P}[H \neq m, D_R = m]
$$

$$
= \sum_{m=0}^{M-1} \mathbb{P}[H \neq m] \mathbb{P}[D_R = m]
$$

$$
= \sum_{m=0}^{M-1} (1 - p_m) \frac{1}{M}
$$

$$
= \frac{1}{M} \sum_{m=0}^{M-1} (1 - p_m)
$$

$$
= \frac{M - 1}{M}
$$

since

$$
\sum_{m=0}^{M-1} (1 - p_m) = M - \sum_{m=0}^{M-1} p_m = M - 1.
$$

The result of this calculation is independent of the prior $p$ on $H$. 