ENEE 621 Spring 2016 Homework 2 Solutions

1. We first discuss the case $y \in [0,5]$

The likelihood ratio can be written as

$$L(y) = \frac{p_1(y)}{p_0(y)} = \frac{\sqrt{2\pi}}{5} e^{\frac{y^2}{2}} \quad , y \in [0,5]$$

The likelihood test ratio is

$$\tau = \frac{\pi_0 (C_{10} - C_{00})}{\pi_1 (C_{01} - C_{11})} = 3$$

If $y \notin [0,5]$, obviously we have to accept H_0.

Therefore, $\Gamma_1 = \{ y \mid y \in [0,5] \ \& \ L(y) \geq 3 \}$

$$= \{ y \mid y \in [\sqrt{2\ln(\tfrac{15}{\sqrt{2\pi}})}, 5] \}$$

The Bayes test is

$$\delta_B(y) = \begin{cases} 1 & \text{if } y \in [\sqrt{2\ln(\tfrac{15}{\sqrt{2\pi}})}, 5] \\ 0 & \text{otherwise} \end{cases}$$

The minimum Bayes risk can be written as

$$r(\delta_B) = \frac{3}{4} \int_{\sqrt{2\ln\frac{5}{\sqrt{2\pi}}}}^{5} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy + \frac{1}{4} \times \frac{\sqrt{2\ln\frac{5}{\sqrt{2\pi}}}}{5}$$

2. Here we have $p_0(y) = p_N(y+s)$ and

$p_1(y) = p_N(y-s)$, which gives

$$L(y) = \frac{1+(y+s)^2}{1+(y-s)^2}$$

With equal priors and uniform costs, the threshold is $\tau = \frac{\pi_0 (C_{10} - C_{00})}{\pi_1 (C_{01} - C_{11})} = 1$.

Therefore, $\Gamma_1 = \{ y \mid L(y) \geq 1 \}$

$= \{ y \mid 1+(y+s)^2 \geq 1+(y-s)^2 \}$

$= \{ y \mid y \geq 0 \}$.

The Bayes test is

$$\delta_B(y) = \begin{cases} 1 & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}$$

The minimum Bayes risk is

$$r(\delta_B) = \frac{1}{2}\int_0^{\infty} \frac{1}{\pi(1+(y+s)^2)}\,dy + \frac{1}{2}\int_{-\infty}^{0} \frac{1}{\pi(1+(y-s)^2)}\,dy$$

$$= \frac{1}{2} - \frac{\tan^{-1}(s)}{\pi}$$

3. The densities under the two hypotheses are:

$$p_0(y) = p(y) = e^{-y}, \quad y > 0$$

and

$$p_1(y) = \int_{-\infty}^{\infty} p(y-s)\, p(y)\, ds = ye^{-y}, \quad y > 0$$

Thus, the likelihood ratio is

$$L(y) = \frac{p_1(y)}{p_0(y)} = y, \quad y > 0$$

4. Since $r_1 \cdots r_k$ are i.i.d, we can have

$$p_0(r_1 \cdots r_k) = \prod_{l=1}^{k} \left(\frac{r_l}{\sigma_0^2} e^{-\frac{r_l^2}{2\sigma_0^2}} \right), \quad r_l > 0$$

$$p_1(r_1 \cdots r_k) = \prod_{l=1}^{k} \left(\frac{r_l}{\sigma_1^2} e^{-\frac{r_l^2}{2\sigma_1^2}} \right), \quad r_l > 0$$

The likelihood ratio is

$$L(r_1 \cdots r_k) = \frac{p_1(r_1 \cdots r_k)}{p_0(r_1 \cdots r_k)} = \left(\frac{\sigma_0^2}{\sigma_1^2}\right)^k e^{\frac{\sum_l r_l^2}{2\sigma_0^2} - \frac{\sum_l r_l^2}{2\sigma_1^2}}, \quad r_l > 0$$

The threshold is

$$\tau = \frac{\pi_0 (C_{10} - C_{00})}{\pi_1 (C_{01} - C_{11})}$$

The likelihood ratio test is

$$\delta(y) = \begin{cases} 1 & \text{if } L(y) \geq \tau \\ 0 & \text{if } L(y) < \tau \end{cases}$$

where $y = [r_1 \cdots r_k]$ and $r_l > 0$ for $l = 1 \cdots k$

5. $P_F(L_{rt\eta}) = P(L(r_1 \cdots r_k) \geq \eta \mid H_0)$

$$= P\left(\left(\frac{\sigma_0^2}{\sigma_1^2}\right)^k e^{\left(\frac{1}{2\sigma_0^2} - \frac{1}{2\sigma_1^2}\right)\sum_i r_i^2} \geq \eta \mid H_0\right)$$

$$= P\left(\sum_i r_i^2 \geq I \mid H_0\right)$$

where $$I = \frac{\ln \frac{\eta \sigma_1^{2k}}{\sigma_0^{2k}}}{\frac{1}{2\sigma_0^2} - \frac{1}{2\sigma_1^2}}$$

Since $r_1 \cdots r_k$ are i.i.d random variables and

$r_i \sim \text{Rayleigh}(\sigma_0)$ under H_0,

$\sum_i r_i^2 \sim \Gamma(k, 2\sigma_0^2)$, which is the Gamma distribution with parameter $(k, 2\sigma_0^2)$

Therefore, $$P_F(L_{rt\eta}) = \int_I^\infty \frac{1}{\Gamma(k)(2\sigma_0^2)^k} x^{k-1} e^{-\frac{x}{2\sigma_0^2}} dx$$

$$= 1 - \frac{1}{\Gamma(k)} \gamma\left(k, \frac{I}{2\sigma_0^2}\right)$$

where $\Gamma(k)$ is the Gamma function,

$\gamma(k, \frac{I}{2\sigma_0^2})$ is the incomplete Gamma function.

$$P_M(L_{rt\eta}) = P(L(r_1 \cdots r_k) < \eta \mid H_1)$$

$$= P\left(\sum_i r_i^2 < T \mid H_1\right)$$

Since $\sum_i r_i^2 \sim \Gamma(k, 2\sigma_1^2)$ under H_1

$$P_M(L_{rt\eta}) = \int_0^T \frac{1}{\Gamma(k)(2\sigma_1^2)^k} x^{k-1} e^{-\frac{x}{2\sigma_1^2}}\, dx$$

$$= \frac{1}{\Gamma(k)} \gamma\left(k, \frac{T}{2\sigma_1^2}\right)$$

6. The likelihood ratio can be written as

$$L(z) = \frac{p_1(z)}{p_0(z)}$$

$$= \frac{f_1(\sqrt{z}) + f_1(-\sqrt{z})}{f_0(\sqrt{z}) + f_0(-\sqrt{z})}, \quad z \geq 0$$

where

$$f_1(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2\sigma^2}}$$

$$f_0(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

The requirement of minimizing the error probability is equivalent to minimizing the Bayes cost with

$$C_{00} = C_{11} = 0, \qquad C_{10} = C_{01} = 1.$$

Therefore, the threshold $\tau = \frac{\pi_0 (C_{10} - C_{00})}{\pi_1 (C_{01} - C_{11})} = 1$

The decision rule is

$$\delta(z) = \begin{cases} 1 & \text{if } L(z) \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

for $z \geq 0$

7. From the decision rule in problem 6, we know that

$$\Gamma_1(z) = \{ z \mid L(z) \geq 1 \}, \quad z \geq 0.$$

By investigating $L(z)$, we find

$$\frac{\partial L(z)}{\partial \sqrt{z}} = \frac{e^{-\frac{2\sqrt{z}+1}{2\sigma^2}} \left(e^{\frac{4\sqrt{z}}{2\sigma^2}} - 1\right)}{2\sigma^2}$$

$$\geq 0 \quad \text{for} \quad z \geq 0.$$

Therefore, $L(z)$ is monotonically increasing for $z \geq 0$.

We find that $L(0) < 1$ and $L(1) > 1$. Therefore, there is $z_t \in (0,1)$ such that $L(z_t) = 1$ and the decision rule is

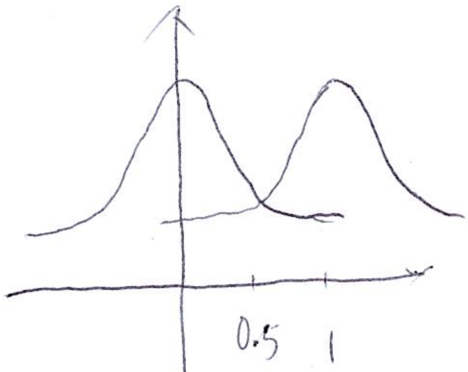
$$\delta(z) = \begin{cases} 1 & \text{if } z \geq z_t \\ 0 & \text{otherwise} \end{cases}$$

We analyze the P_F first.

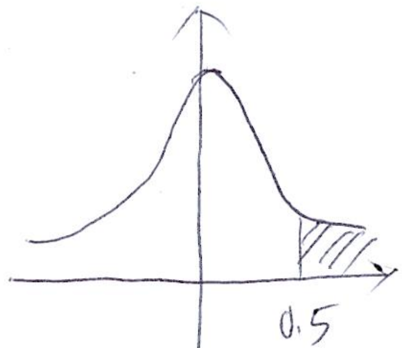
If we use y, then $L(y) = 1$ at $y = 0.5$ as shown in Figure (1), then P_F is shown in Figure (2)

If we use z, according to the decision rule above, P_F is shown in Figure 3.

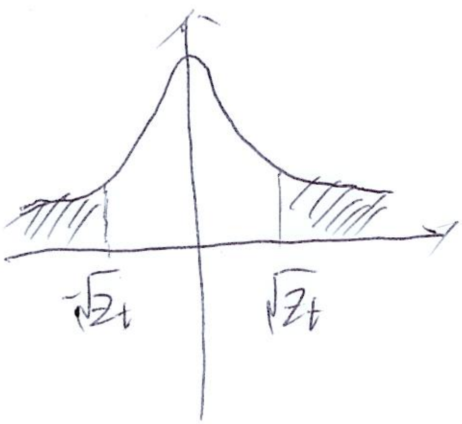
Similarly, the P_m using z is shown in Figure 5, P_m using y is shown in Figure 4.



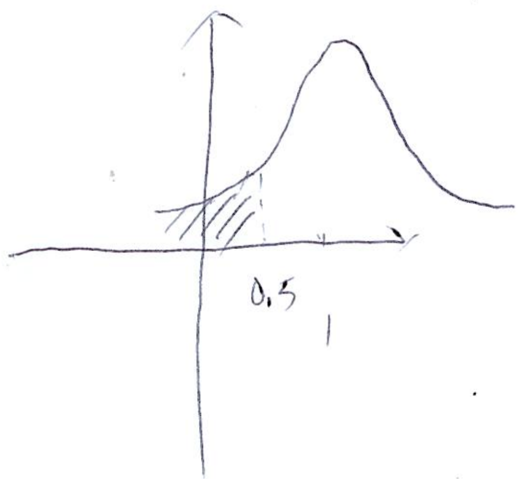
(1)



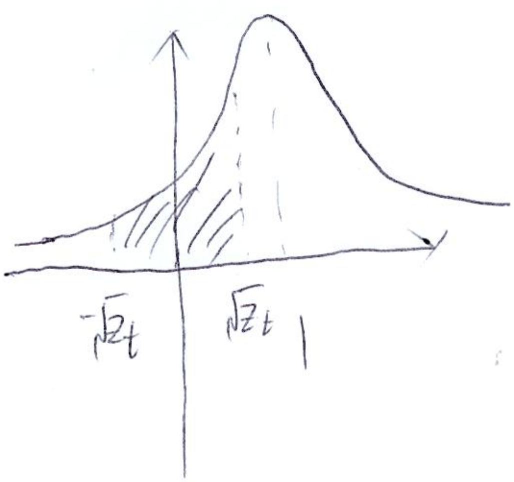
(2)



(3)



(4)



(5)

8. Since $Y_1 \cdots Y_k$ are i.i.d, we can have

$$p_0(Y_1 \cdots Y_k) = \prod_{l=1}^{k} \left(a_0 Y_l + (1-a_0)(1-Y_l)\right), \quad Y_l \in \{0,1\}$$

$$p_1(Y_1 \cdots Y_k) = \prod_{l=1}^{k} \left(a_1 Y_l + (1-a_1)(1-Y_l)\right), \quad Y_l \in \{0,1\}$$

The likelihood ratio can be written as

$$L(Y_1 \cdots Y_k) = \frac{p_1(Y_1 \cdots Y_k)}{p_0(Y_1 \cdots Y_k)}$$

$$= \frac{\prod_{l=1}^{k} \left(a_1 Y_l + (1-a_1)(1-Y_l)\right)}{\prod_{l=1}^{k} \left(a_0 Y_l + (1-a_0)(1-Y_l)\right)}, \quad Y_l \in \{0,1\}$$

The threshold is $T = \dfrac{\pi_0 (C_{10} - C_{00})}{\pi_1 (C_{01} - C_{11})}$

the likelihood ratio test is

$$\delta(Y_1 \cdots Y_k) = \begin{cases} 1 & \text{if } L(Y_1 \cdots Y_k) \geq T \\ 0 & \text{if } L(Y_1 \cdots Y_k) < T \end{cases}$$

for $Y_l \in \{0,1\}$

9. In the context of problem 5,

$$P_F = P\left(\sum_i Y_i^2 \geq T \,\middle|\, H_0\right)$$

where each $Y_i^2 \sim \Gamma(1, 2\sigma_0^2)$ with mean $2\sigma_0^2$ and variance $4\sigma_0^4$

If k is large,

$$\sqrt{k}\left(\frac{1}{k}\sum_i Y_i^2 - 2\sigma_0^2\right)$$ can be approximated by the Gaussian distribution $\mathcal{N}(0, 4\sigma_0^4)$

therefore, $P_F = P\left(\sqrt{k}\left(\frac{1}{k}\sum_i Y_i^2 - 2\sigma_0^2\right) \geq T'\right)$

where $T' = \sqrt{k}\left(\frac{T}{k} - 2\sigma_0^2\right)$

Let $F_0(x; \sigma_0)$ denote the CDF of random variable W_0 where $W_0 \sim \mathcal{N}(0, 4\sigma_0^4)$, then

$$P_F = 1 - F_0(T', \sigma_0)$$

Similarly, let $F_1(x; \sigma_1)$ denote the CDF of random variable W_1 where $W_1 \sim \mathcal{N}(0, 4\sigma_0^4)$, then

$$P_m = F_1(T', \sigma_1)$$