1. Note that $Y_1$ is independent of $\{X_n\}_{n=1}^\infty$. Now, observe that $H_0$ is a composite hypothesis, and we are in a Bayesian situation with the rv $\theta$ (= the rv $P_0$) taking values in $\Theta_0 = \{1/4, 3/4\}$ with $P_0[\theta = 1/4] = \frac{1}{4} = 1 - P_0[\theta = 3/4]$, where $P_0$ denotes the conditional pmf of $\theta$ given $H = H_0$. $H_1$ is a simple hypothesis.

Under $H_0$: For each $t \in \Theta_0 = \{1/4, 3/4\}$, $\{Y_n\}_{n=1}^\infty$ is a 1st order Markov process with transition probabilities:

$$P_t[Y_{n+1} = 0|Y_n = 0] = P_t[X_n = 0|Y_n = 0] = P_t[X_n = 0] \quad (why?)$$

$$= t$$

Similarly,

$$P_t[Y_{n+1} = 0|Y_n = 1] = P_t[Y_{n+1} = 0|Y_n = 1] = 1 - t,$$

and

$$P_t[Y_{n+1} = 1|Y_n = 1] = t.$$ 

Let $Y = (Y_1, \ldots, Y_n)$. Given a sequence of observations $y = (y_1, \ldots, y_N)$ where $P_0[Y_1 = 0] = 1$, we have for each $t \in \{1/4, 3/4\}$: $f_t(y) = P_t[Y = y] = (1 - t)^{\hat{N}}t^{N-1-\hat{N}}$, where $\hat{N}$ is a rv with values in $\{0, 1, \ldots, N - 1\}$ denoting the number of transitions from "0" to "1" and from "1" to "0" in $y = (y_1, \ldots, y_N)$ (with $y_1 = 0$). Then:

$$\hat{f}_0(y) = f_{1/4}(y)P_0[\theta = 1/4] + f_{3/4}(y)P_0[\theta = 3/4]$$

$$= \left(\frac{3}{4}\right)^{\hat{N}}\left(\frac{1}{4}\right)^{N-1-\hat{N}}\left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^{\hat{N}}\left(\frac{3}{4}\right)^{N-1-\hat{N}}\left(\frac{3}{4}\right)$$

$$= \left(\frac{1}{4}\right)^N \left[3^{\hat{N}} + 3^{N-\hat{N}}\right].$$

Under $H_1$: $\{Y_n\}_{n=1}^\infty$ is a 1st order Markov process with transition probabilities

$$P_1[Y_{n+1} = 0|Y_n = 0] = P_1[Y_{n+1} = 0|Y_n = 1] = 1/2$$
Then, for $P[Y_1 = 0] = 1$, we have

$$f_1(y) = P[Y = y] = \left(\frac{1}{2}\right)^{\tilde{N}} \left(\frac{1}{2}\right)^{N-\tilde{N}} = \left(\frac{1}{2}\right)^{N-1}.$$ 

Then:

$$(\ellrt)_\eta : \hat{L}(y) = \frac{f_1(y)}{f_0(y)} \overset{H_0}{\gtrless} \eta$$

i.e.,

$$\left(\frac{1}{4}\right)^{N-1} \overset{H_0}{\lesssim} \eta$$

i.e.,

$$3\tilde{N} + 3^{N-N} \overset{H_0}{\gtrless} \frac{2^{N+1}}{\eta}$$

The LHS is the test statistic, and the RHS is the threshold.

(b) $g_{MAP}(y) = \arg\max_{\theta \in \{1/4, 3/4\}} g_y(\theta)$. Since $g_y(\theta) = \frac{f_0(y)g(\theta)}{f(\theta)}$, it suffices to maximize the numerator as a function of $\theta$.

For $\theta = \frac{1}{4}$, $f_\theta(y)g(\theta) = (1 - \frac{1}{4})^{\tilde{N}} \left(\frac{1}{4}\right)^{N-1-\tilde{N}} \cdot \frac{1}{4} = 3^{\tilde{N}} 4^{-N}$.

For $\theta = \frac{3}{4}$, $f_\theta(y)g(\theta) = (1 - \frac{3}{4})^{\tilde{N}} \left(\frac{3}{4}\right)^{N-1-\tilde{N}} \cdot \frac{3}{4} = 3^{N-\tilde{N}} 4^{-N}$.

$$\Rightarrow g_{MAP}(y) = \begin{cases} \frac{1}{4}, & \text{if } \tilde{N} \geq N - \tilde{N} \text{ i.e. if } \tilde{N} \geq N/2 \\ \frac{3}{4}, & \text{if } \tilde{N} \leq \frac{N}{2} \end{cases}$$

2. Given $c(0,0) = c(1,1) = 0$, and $\nu_0 = c(0,1) = 2c(1,0) = 2\nu_1$. For the Bayes' decision rule $d^* : \mathcal{Y}^* = \{y \in \mathbb{R} : d^*(y) = 0\} = \{y \in \mathbb{R} : h(y) < 0\}$, where $h(y) = \nu_1 p f_1(y) - 2\nu_1 (1-p) f_0(y)$. Clearly for $|y| \geq 1$, $h(y) = \nu_1 p f_1(y) \geq 0 \Rightarrow d^*(y) = 1$ for $|y| \geq 1$.

Next observe that $f_1(y) = 0 \Rightarrow f_0(y) = 0$, so that

$$\ellrt : \frac{f_0(y)}{f_1(y)} \overset{H_0}{\gtrless} \frac{\nu_1 p}{\nu_1 (1-p)} = \frac{p}{1-p}.$$ 

2
We are only concerned with $|y| < 1$ now, in which case

$$\ell_{rt} : \frac{1 - |y|}{2 - |y|} \overset{H_0}{\underset{H_1}{\gtrless}} \frac{1}{8} \cdot \frac{p}{1 - p},$$

which simplifies to $|y|(9p - 8) \overset{H_0}{\underset{H_1}{\gtrless}} (10p - 8)$. so that for $|y| < 1$:

$$|y| \overset{H_0}{\underset{H_1}{\geq}} \frac{10p - 8}{9p - 8}, \text{ if } p \in \left(\frac{8}{9}, 1\right)$$

$$|y| \overset{H_1}{\underset{H_0}{\geq}} \frac{10p - 8}{9p - 8}, \text{ if } p \in \left(0, \frac{8}{10}\right)$$

and always say $H_1$ if $p \in \left[\frac{8}{10}, \frac{8}{9}\right]$.

3. $H_0$ is a simple hypothesis whereas $H_1$ is composite. Fix $p_1 \neq p_0$ and consider the simple hypothesis testing problem that results. Consider the Neyman-Pearson test of size $\alpha$. In a manner similar to Prob. 1, we get:

$$\alpha^{NP(\alpha; p_1)} : \frac{(1 - p_1)\tilde{N}p_{1}^{N-1} - \tilde{N}H_1}{(1 - p_0)\tilde{N}p_{0}^{N-1} - \tilde{N}H_0} \overset{\tilde{N}^{H_1}}{\gtrless} \eta(\alpha; p_1)$$

where $\tilde{N} \in \{0, \ldots, N - 1\}$ is a r.v. representing the # of transitions from “0” to “1” and from “1” to “0”, and $\eta(\alpha; p_1)$ is the corresponding threshold. Upon simplification

$$d^{NP(\alpha; p_1)} : \tilde{N} \log \frac{(1 - p_1)p_0}{(1 - p_0)p_1} \overset{H_1}{\gtrless} \log \eta(\alpha; p_1) + (N - 1) \log \left(\frac{p_0}{p_1}\right)$$

If $p_0 > p_1$: Then by $\frac{(1 - p_1)p_0}{(1 - p_0)p_1} > 0$.

$$d^{NP(\alpha; p_1)} : \tilde{N} \overset{H_1}{\gtrless} \frac{\log \eta(\alpha; p_1) + (N - 1) \log \left(\frac{p_0}{p_1}\right)}{\log(1 - p_1)p_0 - \log(1 - p_0)p_1}$$

with $\eta(\alpha; p_1)$ such that

$$P\left(\tilde{N} \overset{\tilde{N}^{H_1}}{\gtrless} \frac{\log \eta(\alpha; p_1) + (N - 1) \log \left(\frac{p_0}{p_1}\right)}{\log(1 - p_1)p_0 - \log(1 - p_0)p_1} \mid H = 0\right) = \alpha$$
and let $\nu = \frac{\log \eta(\alpha; p_1) + (N - 1) \log \left(\frac{p_0}{p_1}\right)}{\log(1 - p_1)p_0 - \log(1 - p_0)p_1}$

(assuming $\alpha$ is such that solution $\exists$).

Under $H_0$, the statistics of $\tilde{N}$ depend on $p_0$ so that $\nu = \nu(\alpha; p_0)$, i.e., $\nu$ does not depend on $p_1$. Then

$\mathcal{Y}_{d}^{NP(\alpha;p_1)} = \{y \in \{0, 1\}^N : \tilde{N}(y) < \nu(\alpha; p_0)\}.$

If $p_0 < p_1$: Can show since $\log \left(\frac{1-p_0}{1-p_1}p_1\right) < 0$ that $d_{NP(\alpha;p_1)} : \tilde{N} \sim H_0 \lt H_1$ with $\nu(\alpha; p_0)$, where $\nu'$ does not depend on $p_1$

$\Rightarrow \mathcal{Y}_{d}^{NP(\alpha;p_1)} = \{y \in \{0, 1\}^N : \tilde{N}(y) > \nu'(\alpha; p_0)\}.$

If $p_0 = p_1$:

$\mathcal{Y}_{d}^{NP(\alpha;p_1)} = \left\{ \begin{array}{ll}
\{y \in \{0, 1\}^N : \tilde{N}(y) < \nu(\alpha, p_0)\} & \text{if } p_0 > p_1 \\
\{y \in \{0, 1\}^N : \tilde{N}(y) > \nu'(\alpha, p_0)\} & \text{if } p_0 < p_1.
\end{array} \right.$

Clearly, if $\Theta_0 = \{p_0\}, \Theta_1 = \{p_1 \in (0, 1) = p_1 > p_0\}$, a UMP test of size $\alpha$ exists.

If $\Theta_0 = \{p_0\}, \Theta_1 = \{p_1 \in (0, 1); p_1 < p_0\}, \exists$ UMP test of size $\alpha$. If $\Theta_0 = \{p_0\}, \Theta_1 = (0, p_0) \cup (p_0, 1)$, clearly we must know if $p_1 > p_0$ or $p_1 < p_0$ to execute $d_{NP(\alpha;p_1)}$; hence, no UMP exists. When the UMP test does exist, the test statistic = $\tilde{N}$.

4. Fix $\sigma_1^2 \neq \sigma_0^2$. Consider the corresponding simple hypothesis testing problem with the Neyman-Pearson test of size $\alpha$. Recalling that $f_h(y) = \frac{y}{\sigma_h^2} e^{-y^2/\sigma_h^2}, y \geq 0, h = 0, 1$.

$$d_{NP(\alpha;\sigma_1^2)} = \sigma_0^2 e^{-y^2/(\sigma_1^2 - \sigma_0^2)} H_1 \lt H_0 \lt H_1 \eta(\alpha; \sigma_1^2),$$

where the threshold $\eta$ depends on the size $\alpha$ and on $\sigma_1^2$. Upon simplification,

$$d_{NP(\alpha;\sigma_1^2)} = y^2(\sigma_1^2 - \sigma_0^2) H_1 \lt H_0 \left[ \log \eta(\alpha; \sigma_1^2) + \log \left(\frac{\sigma_1^2}{\sigma_0^2}\right) \right] \sigma_0^2 \sigma_1^2.$$
As in problem 3:

(*) If \( \sigma_1^2 > \sigma_0^2 \): \( Y_{dNP(\alpha, \sigma_1^2)} = \{ y \in [0, \infty), y^2 < \nu(\alpha, \sigma_0^2) \} \) where \( \nu(\alpha, \sigma_0^2) \) is the soln of \( P(Y^2 \geq \nu|H = 0) = \alpha \), and does not depend on \( \sigma_1^2 \).

(**) If \( \sigma_0^2 > \sigma_1^2 \): \( Y_{dNP(\alpha, \sigma_1^2)} = \{ y \in [0, \infty) : y^2 > \nu'(\alpha, \sigma_0^2) \} \) where \( \nu'(\alpha, \sigma_0^2) \) (not depending on \( \sigma_1^2 \)) solves \( P(Y^2 \leq \nu'|H = 0) = \alpha \).

(***) If \( \sigma_0^2 \neq \sigma_1^2 \): \( Y_{dNP(\alpha, \sigma_1^2)} = \begin{cases} \{ y \in [0, \infty) : y^2 < \nu(\alpha, \sigma_0^2) \} & \text{if } \sigma_1^2 > \sigma_0^2 \\ \{ y \in [0, \infty) : y^2 > \nu'(\alpha, \sigma_0^2) \} & \text{if } \sigma_0^2 > \sigma_1^2. \end{cases} \)

(a) \( \Theta_0 = \{\sigma_0^2\}, \Theta_1 = (\sigma_0^2, \infty) \Rightarrow \exists \text{ UMP by (*)} \)

(b) \( \Theta_0 = \{\sigma_0^2\}, \Theta_1 = (0, \sigma_0^2) \cup (\sigma_0^2, \infty) \Rightarrow \no \text{ UMP by (***)} \)

(c) \( \Theta_0 = \{\sigma_0^2\}, \Theta_1 = (0, \sigma_0^2) \Rightarrow \exists \text{ UMP by (**).} \)

5. Let \((Y_1, \ldots, Y_N)\) represent \( N \) independent coin tosses with \( Y_i = 1 \) if head, 0 if tail, \( 1 \leq i \leq N \). The LRT is: \( \ell_{rt \eta} : \frac{f_1(Y_1, \ldots, Y_N)_{H_1}}{f_0(Y_1, \ldots, Y_N)_{H_0}} \overset{\eta}{\geq} \eta \)

\[ \Rightarrow \frac{p^{\tilde{N}}(1 - p)^{N - \tilde{N}}}{\left( \frac{1}{2} \right)^N} \overset{H_1}{\underset{H_0}{\geq}} \eta, \]

where \( \tilde{N} = \tilde{N}(Y) \) is a r.v. with values in \( \{0, \ldots, N\} \) and represents the number of heads (note: \( \tilde{N}(Y) = \sum_{i=1}^{N} Y_i \)). Simplifying and using the fact that \( p \in \left( \frac{1}{2}, 1 \right) \), we have

\[ \ell_{rt \eta} : \tilde{N} \overset{H_1}{\underset{H_0}{\geq}} \frac{\log \eta - N \log(2(1 - p))}{\log \left( \frac{p}{1 - p} \right)} \]

Clearly \( S_N = \tilde{N} \). Under each hypothesis, \( \tilde{N} \) is binomial so that

\[ P(\tilde{N} = k|H = 0) = \binom{N}{k} \left( \frac{1}{2} \right)^N, k = 0, \ldots, N, \]

\[ P \left( \tilde{N} = k|H = 1 \right) = \binom{N}{k} p^k (1 - p)^{N-k}, k = 0, \ldots, N. \]

\[ p_F \left( d^{NP(\alpha)} \right) = \alpha \iff P(N \geq \nu|H = 0) = \alpha, \]
where $\nu$ solves

$$\sum_{k=[\nu]}^{N} P(\tilde{N} = k | H = 0) = \alpha$$

assuming $\alpha$ is such that soln. exists

$$\Rightarrow \left( \frac{1}{2} \right)^N \sum_{k=[\nu]}^{N} \binom{N}{k} = \alpha \Rightarrow [\nu] = [\nu](\alpha), \text{ a fn of } \alpha,$$

$$\Rightarrow d^{NP(\alpha)} : \tilde{N} \overset{H_1}{\underset{H_0}{\succ}} [\nu].$$

6.

$$H_0 : Y_t = N_t$$

$$H_1 : Y_t = s_t + N_t \quad t = 1, \ldots, K.$$  

The $K \times K$-covariance matrix $R_K$ for the noise process $\{N_t, t = 1, \ldots, k\}$ has entries $R_K(t, \tau) = t \wedge \tau, 1 \leq t, \tau \leq K$. Observe that

$$R_K = \begin{bmatrix}
1 & 1 & \ldots & 1 \\
1 & 2 & 2 & \ldots & 2 \\
1 & 2 & 3 & \ldots & 3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 2 & 3 & 4 & \ldots & K
\end{bmatrix}$$

It can be shown by induction that $\det R_K = 1$ (clearly $\det R_1 = \det R_2 = 1$).

(a) From class notes:

$$d^{NP(\alpha)} : y^T R_K^{-1} s \overset{H_1}{\underset{H_0}{\succ}} \nu(\alpha)$$

where $y^T = (y_1, \ldots, y_k), s = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ with $K$ elements. Now, $R_K^{-1} R_K = I_{K \times K}$; $s =$
\[
\begin{bmatrix}
1 \\
1 \\
\vdots \\
1 \\
\end{bmatrix}
\]
is the 1st column of \( R_K \) so that \( R_K^{-1} s = 1 \)st column of \( I_{K \times K} \).

\[
y^T R_K^{-1} s = y_1 \Rightarrow d^{NP(\alpha)} : y_1 \overset{H_0}{\geq} \nu(\alpha)
\]

where \( \nu = \nu(\alpha) \) solves: \( P(Y_1 > \nu|H = 0) = \alpha \) i.e., \( 1 - \Phi(\nu) = \alpha \), or \( \Phi(\nu) = 1 - \alpha \Rightarrow \nu = x_{1-\alpha} \) (a function only of \( \alpha \); see example of composite hypothesis testing in class notes for notation: \( x_\alpha = \Phi^{-1}(\alpha) \)). Then,

\[
p_D(d^{NP(\alpha)}) = P(Y_1 \geq \nu|H = 1)
\]

\[
= P(Y_1 - 1 \geq \nu - 1|H = 1), \text{ where } Y_1 - 1 \sim N(0,1)
\]

\[
= 1 - \Phi(\nu - 1) = 1 - \Phi(x_{1-\alpha} - 1)
\]

(b) \( d^{NP(\alpha)} : y^T R_K^{-1} s \overset{H_0}{\geq} \nu(\alpha) \), where \( s = \begin{bmatrix} 1 \\
2 \\
\vdots \\
k \end{bmatrix} \) is also the last column of \( R_K \) so that \( R_K^{-1} s = \) last column of \( I_{K \times K} \Rightarrow y^T R_K^{-1} s = y_K \). Then proceed in a manner similar to part (a). (Note: \( E[N^2_K] = K \).)

7. (a) Let a “head” be the event 1, and a “tail” the event 0. Fix \( \theta \) in \((0,1),\theta \neq 1/2\), and consider the following simply hypothesis testing problem. (The original problem is one of composite hypothesis testing which we shall get to shortly.)

\[
H_0 : \{X_i\}^{N \text{i.i.d.}}, P(X_i = 1|H = 0) = \frac{1}{2}; \{Y_i\}^{N \text{i.i.d.}}, P(Y_i = 1|H = 0) = \theta
\]

\[
H_1 : \{X_i\}^{N \text{i.i.d.}}, P(X_i = 1|H = 0) = \theta; \{Y_i\}^{N \text{i.i.d.}}, P(Y_i = 1|H = 0) = 1/2.
\]

(Thus, \( H_0 \) says the X-coin is fair, the Y-coin is biased with bias = fixed prob. \( \theta \neq 1/2 \); \( H_1 \) says that the Y-coin is fair, the X-coin is bias “\( \theta \)”.) Let \( N_X = \) a r.v. in \( \{0;\ldots,N\} \) denoting the # of heads of the X-coin, and \( N_Y = \) a r.v. in \( \{0,\ldots,N\} \) denoting the
number of heads of the \( Y \)-coin. Observe that since \( \{X_i\}_{1}^{N} \) is independent of \( \{Y_i\}_{1}^{N} \), we have \( N_X \) independent of \( N_Y \).

\[
\ell_{\text{lrt}}: L(x,y) \xrightarrow{H_1}{H_0} \eta, \quad x = (x_1, \ldots, x_N) \quad y = (y_1, \ldots, y_N). 
\]

where

\[
L(x,y) = \frac{\theta^{N_X} (1 - \theta)^{N - N_X} \left(\frac{1}{2}\right)^N}{(\frac{1}{2})^N \theta^{N_Y} (1 - \theta)^{N - N_Y}}
\]

\[
\Rightarrow \ell_{\text{lrt}}: \frac{\theta^{N_X} (1 - \theta)^{N - N_X}}{\theta^{N_Y} (1 - \theta)^{N - N_Y}} \xrightarrow{H_1}{H_0} \eta. 
\]

Now consider the composite hypothesis testing problem with a view to setting up the generalized \( \ell_{\text{rt}} \). Observe that

\[
\Theta_0 = \{1/2\} \times \{1/2\}^c \quad \text{where} \quad \{1/2\}^c \triangleq (0,1/2) \cup (1/2,1)
\]

(so that \( \theta_0 = (1/2, \theta) \) a pair of parameters, where \( \theta \in, \{1/2\}^c \)). Likewise \( \Theta_1 = \{1/2\}^c \times \{1/2\} \), so that \( \theta_1 = (\theta, 1/2) \). Then the generalized LRT is:

\[
\begin{align*}
\text{g} \ell_{\text{lrt}}: \hat{L}(x,y) &= \max_{\theta \in (1/2)^c} \frac{\theta^{N_X} (1 - \theta)^{N - N_X}}{\max_{\theta \in (1/2)^c} \theta^{N_Y} (1 - \theta)^{N - N_Y}} . \\
\end{align*}
\]

It is easily verified that the maximizing values of \( \theta \) are:

\[
\hat{\theta}(N_X) = \frac{N_X}{N} \text{ in the numerator}
\]
\[
\hat{\theta}(N_Y) = \frac{N_Y}{N} \text{ in the denominator}
\]

both should differ from 1/2 for \( \text{g} \ell_{\text{lrt}} \) to exist.

\[
\Rightarrow \text{g} \ell_{\text{lrt}}: \hat{L}(x,y) = \frac{\left(\frac{N_X}{N}\right)^{N_X} \left(\frac{N - N_X}{N}\right)^{N - N_X}}{\left(\frac{N_Y}{N}\right)^{N_Y} \left(\frac{N - N_Y}{N}\right)^{N - N_Y}} \xrightarrow{H_1}{H_0} \eta
\]

which simplifies to:

\[
\frac{N_X^{N_X} (N - N_X)^{N - N_X}}{N_Y^{N_Y} (N - N_Y)^{N - N_Y}} \xrightarrow{H_1}{H_0} \eta
\]
Next, to check for conditions for a UMP to exist: for a fixed \( \theta \) in \( \{1/2\}^c \), the Neyman-Pearson test of size \( \alpha \) is:

\[
d_{NP}(\alpha; \theta) : \theta^N_X - N_Y (1 - \theta)^N_Y - N_X \begin{cases} \geq & H_1 \\ < & H_0 \end{cases}
\]

i.e.,

\[
\left( \frac{\theta}{1 - \theta} \right)^{N_X - N_Y} \begin{cases} \geq & H_1 \\ < & H_0 \end{cases} \eta(\alpha, \theta),
\]

i.e.,

\[(N_X - N_Y) \log \left( \frac{\theta}{1 - \theta} \right) \begin{cases} \geq & H_1 \\ < & H_0 \end{cases} \log \eta(\alpha, \theta).
\]

Then as in probs. 3, 4;

($) If \( \theta \in (1/2, 1) : d_{NP}(\alpha; \theta) : N_X - N_Y \begin{cases} \geq & H_1 \\ < & H_0 \end{cases} \nu(\alpha), \) where \( \nu \) solves:

\[
P(N_X - N_Y \geq \nu|H = 0) = \alpha \text{ (assuming soln. } \exists).\]

(*) If \( \theta \in (0, 1/2) : d_{NP}(\alpha; \theta) : N_Y - N_X \begin{cases} \geq & H_1 \\ < & H_0 \end{cases} \nu'(\alpha), \) where \( \nu' \) solves

\[
P(N_Y - N_X \geq \nu'|H = 0) = \alpha \text{ (assuming soln. } \exists).\]

Now, if \( \Theta_0 = \left\{ \frac{1}{2} \right\} \times \left( \frac{1}{2}, 1 \right), \Theta_1 = \left( \frac{1}{2}, 1 \right) \times \left\{ \frac{1}{2} \right\} \), UMP test exists as ($) obtains. If \( \Theta_0 = \left\{ \frac{1}{2} \right\} \times (0, \frac{1}{2}), \Theta_1 = \left( \frac{1}{2}, \frac{1}{2} \right) \times \left\{ \frac{1}{2} \right\}, \exists \) UMP since (*) obtains. As in probs 3,4, if \( \Theta_0 = \left\{ \frac{1}{2} \right\} \times \left\{ \frac{1}{2} \right\}^c, \Theta_1 = \left\{ \frac{1}{2} \right\}^c \times \left\{ \frac{1}{2} \right\}, \) \( \not\exists \) UMP. The test statistic in the first two cases is \( N_X - N_Y \). item To compute \( p_F \), let us consider the case \( \Theta_0 = \left\{ \frac{1}{2} \right\} \times (1/2, 1), \Theta_1 = (1/2, 1) \times \left\{ \frac{1}{2} \right\} \). For the MPE criterion, \( \eta(\alpha, \theta) = 1 \) so that

\[
\ell_{rt} : N_X - N_Y \begin{cases} \geq & H_1 \\ < & H_0 \end{cases} \]

\[
\Rightarrow p_F(d^*) \triangleq \sup_{\theta \in (1/2, 1)} p_F(d^*_\theta)
\]
i.e.,

\[ p_F(d^*) = \sup_{\theta \in (\frac{1}{2}, 1)} P(N_X \geq N_Y | H = 0) \]

\[ = \sup_{\theta \in (\frac{1}{2}, 1)} \sum_{k=0}^{N} P(N_X \geq k | N_Y = k, H = 0) P(N_Y = k | H = 0) \]

\[ = \sup_{\theta \in (\frac{1}{2}, 1)} \sum_{k=0}^{N} P(N_X \geq k | H = 0) P(N_Y = k | H = 0) \]

recall: \( N_X \) is independent of \( N_Y \) under \( H_0 \) and \( H_1 \).

\[ = \sup_{\theta \in (\frac{1}{2}, 1)} \sum_{k=0}^{N} \left( \sum_{i=k}^{N} \binom{N}{i} \left( \frac{1}{2} \right)^N \right) \left( \binom{N}{k} \theta^k (1 - \theta)^{N-k} \right) \]

ENJOY IT!