1. Let $\theta$ be a $\mathbb{R}$-valued Gaussian r.v. with mean 1 and variance 2. Two observations $Y_1$ and $Y_2$ of $\theta$ in noise are made as follows:

$$Y_1 = (1 + N_1)\theta, \quad Y_2 = (1 + N_2)\theta,$$

where $\theta$ is independent of $(N_1, N_2)$, $E[N_1] = E[N_2] = 0$, $E[N_1^2] = E[N_2^2] = 1$, and $N_1 \perp N_2$.

(i) Find $\hat{E}[\theta | Y_1]$.

(ii) Find $\hat{E}[\theta | Y_1, Y_2]$ in terms of $\hat{E}[\theta | Y_1]$ and $Y_2$ in a recursive fashion.

2. Consider the one-dimensional system

$$\theta_{t+1} = \theta_t + bW_t, \quad t = 0, 1, \ldots$$

where the observation process is

$$Y_t = \theta_t + W_t, \quad t = 0, 1, \ldots$$

Here, $\{W_t\}_{t=0}^{\infty}$ is a sequence of $\mathbb{R}$-valued Gaussian r.v.’s with $E[W_t] = 0$, $E[W_tW_s] = Q_{ts}$, where $Q_t > 0$, $t, s = 0, 1, \ldots$. Let $\theta_0 \sim \mathcal{N}(0, \sum_0)$, and assume that $\theta_0 \perp \{W_t\}_{t=0}^{\infty}$; $b \neq 0$ is a known constant. Let $Y^t \triangleq (Y_0, \ldots, Y_t)$.

(a) Find a recursive scheme for generating $\hat{E}[\theta_t | Y^t], t = 0, 1, \ldots$.

(b) Find a recursive scheme for generating $\hat{E}[\theta_{t+T} | Y^t]$, where $T$ is a positive integer.

3. Let $\theta, Y$ and $Z$ be $\mathbb{R}$-valued r.v.’s with $E[|\theta|^2] < \infty$, $E[|Y|^2] < \infty$, $E[|Z|^2] < \infty$. It is true that

$$\hat{E}[\theta | Y, Z] = \hat{E}[\theta | Y], \hat{E}[Y | Z].$$

(If true, provide a formal proof; if not, give a counter-example.)

4. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of $\mathbb{R}$-valued, zero-mean r.v.’s with $E[X_nX_m] = R_{n-m}$ for all $n, m \geq 1$. Assuming that $R_{n-m} > 0$ for all $n, m \geq 1$, and that $R_0 \neq R_2$, find $\hat{E}[X_n | X_{n-1}, X_{n+1}]$ – the best linear interpolator.
5. Let \( Y^n \triangleq (Y_1, \ldots, Y_n) \) be a sequence of i.i.d. \( \mathbb{R} \)-valued r.v.’s, each distributed uniformly on \([0, \theta]\), where \( \theta \) is itself a \( \mathbb{R} \)-valued r.v. distributed uniformly on \([a, b]\), where \( a, b \) are known constants with \( 0 < a < b \).

Let \( u(\cdot) \) denote the unit step function on \( \mathbb{R} \), i.e.,

\[
u(x) = \begin{cases} 
1, & x \geq 0 \\
0, & \text{otherwise}
\end{cases}
\]

(i) Show that the conditional probability density function of \( \theta \) given \( Y^n = y^n \), where \( y^n = (y_1, \ldots, y_n) \), is

\[
\gamma_{y^n}(t) = \frac{(n-1)t^{-n}u(t - \max_{1 \leq i \leq n} y_i)}{\left[ \max(a, \max_{1 \leq i \leq n} y_i) \right]^{-(n-1)} - b^{-(n-1)}}
\]

(ii) Find the MAP estimate of \( \theta \) given \( Y^n = y^n \).

(iii) For the estimator of part (ii) above, compute the bias \( E[g_{MAP}(Y^n) - \theta] \).

(iv) Show that the minimum mean-squared error estimate of \( \theta \) given \( Y^n = y^n \) is

\[
g_{MSE}(y^n) = \frac{(n-1)}{(n-2)} \left[ \max(a, \max_{1 \leq i \leq n} y_i) \right]^{-(n-2)} - b^{-(n-2)}
\]

(v) Let \( a = 1, b = 2 \). Find the linear least-squares error estimate \( \hat{E}[\theta|Y_1] \) of \( \theta \) given the single observation \( Y_1 \).