1. Let \( T : \mathbb{R}^k \to \mathbb{R}^d \) be a sufficient statistic for the family of probability distributions \( \{F_\theta, \theta \in \Theta\} \) on \( \mathbb{R}^k \). Thus, for each \( \theta \) in \( \Theta \),

\[
P_\theta [Y \in B|T(Y)] = \nu(B, T(Y)), \quad B \in \mathcal{B}(\mathbb{R}^k) \quad P_\theta - a.s.,
\]

for some mapping \( \nu : \mathcal{B}(\mathbb{R}^k) \times \mathbb{R}^d \to [0, 1] \). Consider the Bayesian situation in which \( \theta \) is a \( \Theta \)-valued rv with probability distribution \( G \). With \( P \) being the probability measure on \( \Theta \times \mathbb{R}^k \) as defined in class, show that

\[
P [Y \in B|T(Y), \theta] = \nu'(B, T(Y)), \quad B \in \mathcal{B}(\mathbb{R}^k) \quad P - a.s.,
\]

for some mapping \( \nu' : \mathcal{B}(\mathbb{R}^k) \times \mathbb{R}^d \to [0, 1] \).

2. Consider a family of probability distributions \( \{F_\theta, \theta \in \Theta\} \) on \( \mathbb{R}^k \). For a statistic \( T : \mathbb{R}^k \to \mathbb{R}^d \) and a Borel mapping \( \varphi : \mathbb{R}^d \to \mathbb{R}^\ell \), show that:

(i) if \( \varphi_o T \) is a sufficient statistic for \( \{F_\theta, \theta \in \Theta\} \), then so is \( T \);

(ii) if \( T \) is a sufficient statistic for \( \{F_\theta, \theta \in \Theta\} \), then so is \( \varphi_o T \) provided the mapping \( \varphi \) is invertible.

3. Show that the family of Poisson distributions \( \{P(\theta), \theta > 0\} \) is a complete family.

4. Show that the family of Gaussian distributions \( \{N(\theta, R), \theta \in \mathbb{R}^k\} \) is a complete family.

5. Consider the family of Gaussian distributions \( \{N(0, \theta), \theta > 0\} \) on \( \mathbb{R} \). Determine whether this family is complete.

6. Consider the family of uniform distributions \( \{U(-\theta, \theta), \theta > 0\} \). Is this family complete?

7. Consider the family of distributions \( \{Ber^{(n)}(\theta), \theta \in (0, 1)\} \) corresponding to \( n \) i.i.d. observations of a Bernoulli rv. Show that this family is not complete if \( n \geq 2 \).
8. Consider an estimation problem in which the observations \( \{Y_n, n = 1, 2, \ldots \} \) are \( \mathbb{R} \)-valued rvs given by

\[
Y_n = \theta X + V_n, \quad n = 1, 2, \ldots
\]

where \( \theta \) is in \( \mathbb{R} \), and the rvs \( \{X, V_n, n = 1, 2, \ldots \} \) are i.i.d. \( \mathcal{N}(0, 1) \). Find a sufficient statistic for \( \theta \) (on the basis of \( Y^n \)).

9. For each \( \theta \neq 0 \) in \( \mathbb{R} \), let \((Y_1, \ldots, Y_n), n = 1, 2, \ldots, \) be i.i.d. \( \mathbb{R} \)-valued \( \sim \mathcal{N}(\theta, \theta^2) \) rv’s.

(i) Find a nontrivial sufficient statistic for estimating the parameter \((\theta, \theta^2)\) on the basis of \( Y^n = (Y_1, \ldots, Y_n) \).

(ii) Determine whether the statistic in part (i) above is a complete sufficient statistic.

10. We are interested in estimating a parameter \( \theta > 0 \) on the basis of \( n \) i.i.d. observations, \( Y_1, \ldots, Y_n, n = 1, 2, \ldots, \) each of which is uniformly distributed on the interval \((0, \theta)\).

(i) For the associated family of distributions \( \{U^{(n)}(0, \theta), \theta > 0\} \), find a nontrivial sufficient statistic which is also complete.

(ii) Determine a MVUE for \( \theta \) on the basis of \( Y^n = (Y_1, \ldots, Y_n) \).

11. Consider the \( \mathbb{R} \)-valued observation rv \( Y \) given by

\[
Y = \alpha X + N, \quad \alpha \neq 0
\]

where \( X \) and \( N \) are independent Gaussian rvs, each with mean zero and unit variance, and \( \alpha \) is a nonzero scalar. The parameter to be estimated is \( \theta = \alpha^2 \). Determine whether the family of distributions \( \{F_\theta, \theta > 0\} \) of the rv \( Y \) is complete.