Problem 1

[1a] Since

\[
\frac{\partial}{\partial \theta} \ln f_\theta(y) = \frac{2(y - \theta)}{2\sigma^2},
\]

we have

\[
M(\theta) = \mathbb{E}_\theta \left[ (Y - \theta) \right] = \frac{\sigma^2}{\sigma^4} = \frac{1}{\sigma^2}.
\]

[1b] The ML estimate of \( \theta \) on the basis of \( y^n = (y_1, \cdots, y_n) \) is \( g_{ML}(y^n) = \frac{\sum_{i=1}^n y_i}{n} \). The error covariance of \( g_{ML} \) is

\[
\Sigma_\theta(g_{ML}) = \mathbb{E}_\theta \left[ (g_{ML}(Y^n) - \theta)^2 \right]
\]

\[
= \mathbb{E}_\theta \left[ \left( \frac{1}{n} \sum_{i=1}^n Y_i - \theta \right)^2 \right]
\]

\[
= \frac{1}{n^2} \mathbb{E}_\theta \left[ \left( \sum_{i=1}^n Y_i - n\theta \right)^2 \right]
\]

\[
= \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}.
\]

Noting that \( M^{(n)}(\theta) = \frac{n}{\sigma^2} \), it follows that \( g_{ML} \) is efficient since \( \Sigma_\theta(g_{ML}) = M^{(n)}(\theta)^{-1} \).

Alternatively, the efficiency of \( g_{ML} \) also follows upon verifying that

\[
g_{ML}(y^n) - \theta = M^{(n)}(\theta)^{-1} \frac{\partial}{\partial \theta} f_\theta(y^n).
\]
Problem 2

[2a] Since

\[ f_\theta(y_1, \ldots, y_n) = \prod_{i=1}^{n} \frac{e^{-\theta y_i}}{y_i!} = \frac{e^{-n\theta \sum_{i=1}^{n} y_i}}{\prod_{i=1}^{n} y_i!}, \]

we have for \( \tilde{\theta} \triangleq e^{-\theta} \) that

\[ f_{\tilde{\theta}}(y^n) = \frac{\tilde{\theta}^n \left( \ln \frac{1}{\tilde{\theta}} \right)^{\sum_{i=1}^{n} y_i}}{\prod_{i=1}^{n} y_i!}, \]

where \( \tilde{\theta} \in (0, 1) \).

Next,

\[ \ln f_{\tilde{\theta}}(y^n) = n \ln \tilde{\theta} + \sum_{i=1}^{n} y_i \ln \left( \ln \frac{1}{\tilde{\theta}} \right) - \sum_{i=1}^{n} \ln y_i!, \]

so that

\[ \frac{\partial}{\partial \tilde{\theta}} \ln f_{\tilde{\theta}}(y^n) = \frac{n}{\tilde{\theta}} + \left( \sum_{i=1}^{n} y_i \right) \frac{1}{(\ln \frac{1}{\tilde{\theta}})} \left( -\frac{1}{\tilde{\theta}} \right), \]

which, upon being equated to 0, gives \( \tilde{\theta} = e^{-\frac{\sum_{i=1}^{n} y_i}{n}} \).

Hence, \( \tilde{\theta}_{ML}(y^n) = e^{-\frac{\sum_{i=1}^{n} y_i}{n}} \), and exists provided \( \sum_{i=1}^{n} y_i \neq 0 \).

[2b] Since \( e^{-\theta} = P_\theta(Y_1 = 0) \), we see that the estimator \( g \) defined by

\[ g(y^n) = 1(y_1 = 0) \]

is an unbiased estimator of \( e^{-\theta} \) as

\[ E_\theta [g(Y^n)] = P_\theta(Y_1 = 0) = e^{-\theta}. \]

Next, we know that the statistic \( T_n \) defined by

\[ T_n(y^n) \triangleq \sum_{i=1}^{n} y_i \]

is a sufficient statistic for \( \theta \) and, hence, also for \( e^{-\theta} \). Also, since \( \sum_{i=1}^{n} Y_i \) is a Poisson rv with mean \( n\theta \), \( T_n \) is a complete sufficient statistic for \( \theta \) and, hence, also for \( e^{-\theta} \). Consequently, the
estimator \( \tilde{g}_0 T_n \) is a MVUE for \( e^{-\theta} \), where

\[
\tilde{g}(t) = \mathbb{E}_\theta [g(Y^n)|T_n = t]
\]

\[
= P_\theta(Y_1 = 0|\sum_{i=1}^{n} Y_i = t)
\]

\[
= \frac{P_\theta(Y_1 = 0) P_\theta(\sum_{i=2}^{n} Y_i = t)}{P_\theta(\sum_{i=1}^{n} Y_i = t)}
\]

\[
= \frac{e^{-\theta} e^{-(n-1)\theta} (t)^{t}}{e^{-n\theta} (n\theta)^{t} t!}
\]

\[
= \left( 1 - \frac{1}{n} \right)^t.
\]

Hence, the desired MVUE is

\[
(\tilde{g}_0 T_n)(y^n) = \left( 1 - \frac{1}{n} \right)^{\sum_{i=1}^{n} y_i}.
\]
Problem 3

[3a] We have

\[ M(\theta) = E_\theta \left[ \left| \frac{\partial}{\partial \theta} \ln f_\theta(Y) \right|^2 \right] \]
\[ = E_\theta \left[ \frac{1}{\theta} - Y \right]^2 = \frac{1}{\theta^2}. \]

[3b] Let \( A \) denote the observed event. Then,

\[ P_\theta(A) = \binom{M}{m} (P_\theta(Y > y_o))^m (P_\theta(Y \leq y_o))^{M-m} \]
\[ = \binom{M}{m} (e^{-\theta y_o})^m (1 - e^{-\theta y_o})^{M-m} \]
whence,

\[ \ln P_\theta(A) = \ln \binom{M}{m} - \theta y_o m + (M - m) \ln (1 - e^{-\theta y_o}). \]

Upon setting \( \frac{\partial}{\partial \theta} \ln P_\theta(A) = 0 \), we get

\[ \theta = \frac{1}{y_o} \ln \frac{M}{m}. \]

Thus, \( \theta_{ML}(M, m, y_o) = \frac{1}{y_o} \ln \frac{M}{m} \), which exists provided \( m < M \).