Problem 2.1
Suppose \( x \) and \( y \) are statistically independent random variables with density functions

\[
p_x(x) = \frac{1}{2} \delta(x + 1) + \frac{1}{2} \delta(x - 1),
\]

and

\[
p_y(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{y^2}{2\sigma^2}\right).
\]

Let \( z = x + y \) and \( w = xy \).

(a) Find \( p_z(z) \) the probability density function for \( z \).

(b) Find the conditional probability density functions \( p_{z|x}(z|x = -1) \) and \( p_{z|x}(z|x = +1) \).

(c) Find the mean values \( \bar{x} = m_x \) and \( \bar{y} = m_y \), the variances \( \sigma_x^2 \) and \( \sigma_w^2 \), and the covariance \( \lambda_{yw} \). Are \( y \) and \( w \) uncorrelated random variables? Are \( y \) and \( w \) statistically independent random variables?

(d) Are \( y \) and \( w \) Gaussian random variables? Are they jointly Gaussian? Explain.

Problem 2.2
Let \( \mathbf{x} = [x_1 \ x_2]^T \) be a zero-mean Gaussian random vector with covariance matrix

\[
\mathbf{\Lambda}_x = \begin{bmatrix} 34 & 12 \\ 12 & 41 \end{bmatrix}.
\]

(a) Verify that \( \mathbf{\Lambda}_x \) is a valid covariance matrix.

(b) Find the marginal probability density function for \( x_1 \). Find the probability density function for \( y = 2x_1 + x_2 \).
(c) Find a linear transformation defining a new random vector
\[ x' = \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = P \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]
so that \( x'_1 \) and \( x'_2 \) are statistically independent and so that
\[ PP^T = I, \]
where \( I \) is the 2 \( \times \) 2 identity matrix.

**Problem 2.3**

Let \( \{x_n\}_{n=1}^{\infty} \) be a sequence of random variables with identical means \( \bar{x}_i = m \), and identical variances \( \text{var} x_i = \sigma^2 \). We define the sample mean and the sample mean-square of the first \( N \) of the \( x_i \)'s as follows

\[
\hat{m}_N = \frac{1}{N} \sum_{i=1}^{N} x_i \quad \text{sample mean};
\]
\[
\hat{\sigma}^2_N = \frac{1}{N} \sum_{i=1}^{N} x_i^2 \quad \text{sample mean-square}.
\]

(a) Suppose that the \( x_i \)'s are uncorrelated random variables. Find the mean and the variance of the sample mean. Show that
\[
\lim_{N \to \infty} E \left[ (\hat{m}_N - m)^2 \right] = 0,
\]
and use this result to deduce that
\[
\lim_{N \to \infty} \Pr[|\hat{m}_N - m| > \epsilon] = 0, \quad \text{for any } \epsilon > 0.
\]

(b) Suppose that the \( x_i \)'s are statistically independent zero-mean Gaussian random variables. Find the mean and variance of the sample mean-square. Show that
\[
\lim_{N \to \infty} E \left[ (\hat{\sigma}_N^2 - \sigma^2)^2 \right] = 0.
\]

(c) Suppose that the \( x_i \)'s are statistically independent zero-mean Gaussian random variables. Are \( \hat{m}_N \) and \( \hat{\sigma}_N^2 \) Gaussian random variables? Explain.
Problem 2.4
Let \( \mathbf{x} \) be an \( N \)-dimensional zero-mean random vector whose covariance matrix has eigenvalues
\[ \lambda_1 > \lambda_2 > \cdots > \lambda_N , \]
and corresponding eigenvectors
\[ \phi_1, \phi_2, \cdots, \phi_N . \]
Suppose we wish to approximate \( \mathbf{x} \) as a (scalar) random variable \( b \) times a deterministic \( N \)-dimensional vector \( a \) (i.e., \( b \mathbf{a} \)). We are interested in finding the “best” \( b \) and \( a \), so that \( b \mathbf{a} \) is as “close” (in some appropriate sense) to \( \mathbf{x} \) as possible.

(a) Given a realization \( \mathbf{x} = \mathbf{x} \), determine
\[ b_{\text{opt}} = \arg \min_{b \in \mathbb{R}} ||\mathbf{x} - b \mathbf{a}|| , \]
where \( ||\cdot|| \) is the Euclidean norm on \( \mathbb{R}^N \), i.e., \( ||\mathbf{z}|| = \mathbf{z}^T \mathbf{z} \). Find an explicit expression for \( b_{\text{opt}}(\mathbf{x}) \) in terms of \( \mathbf{x} \) and \( \mathbf{a} \). Note that, with \( \mathbf{x} \) viewed as a random variable, the optimal \( b_{\text{opt}} \) is also a random variable.

(b) Let \( b_{\text{opt}} \) be as defined in part (a). Now define
\[ a_{\text{opt}} = \arg \min_{\mathbf{a} \in \mathbb{R}^N} E \left[ ||\mathbf{x} - b_{\text{opt}} \mathbf{a}||^2 \right] . \]
Show that
\[ a_{\text{opt}} = \arg \max_{\mathbf{a} \in \mathbb{R}^N} \frac{\text{var} \mathbf{a}^T \mathbf{x}}{\mathbf{a}^T \mathbf{a}} . \]

(c) Determine
\[ \max_{\mathbf{a} \in \mathbb{R}^N} \frac{\text{var} \mathbf{a}^T \mathbf{x}}{\mathbf{a}^T \mathbf{a}} \]
and indicate the value(s) of \( \mathbf{a} \) for which the maximum is achieved.

(d) Repeat part (c) when we impose the constraint that
\[ \mathbf{a} \perp \phi_i \quad (i.e., \mathbf{a}^T \phi_i = 0 \text{ for } i = 1, 2, \cdots, k - 1), \]
for some \( k \geq 2 \). Again, indicate the value(s) of \( \mathbf{a} \) for which the maximum is achieved.
Problem 2.5
Let \( V \) denote a general inner-product space and let \( x, y \) denote elements of \( V \). Suppose we want to approximate \( x \) using a multiple of \( y \). That is, let \( \hat{x} = ay \); we want to find \( a \in \mathbb{R} \) so that \( \hat{x} \) is, in some sense, as close as possible to \( x \).

(a) Let \( e = x - \hat{x} = x - ay \). Show that
\[
J = ||e||
\]
is minimized over all possible values of \( a \) when
\[
< e, y >= 0
\]
(b) Find an explicit formula for \( a \) in terms of inner products involving \( x \) and \( y \).
(c) Give explicit formulas for \( a \) in the following two cases:
   (i) \( V = L^2(\mathbb{R}) \).
   (ii) \( V = L^2(\Omega) \). What is \( J \) in this case? (Do not leave your expressions of \( a \) and \( J \) in terms of \( < \cdot, \cdot > \)).

Problem 2.6
Let \( x, y, \) and \( z \) be zero-mean unit-variance random variables satisfying
\[
\text{var}[x + y + z] = 0
\]
Determine the covariance matrix of \( x, y, \) and \( z \); i.e., determine the matrix
\[
\begin{bmatrix}
E[x^2] & E[xy] & E[xz] \\
E[yz] & E[y^2] & E[yz] \\
E[xz] & E[zy] & E[z^2]
\end{bmatrix}
\]
Hint: Use vector space concepts.

Problem 2.7 (optional)
Let \( \{x_n\}_{n=1}^{\infty} \) be a sequence of mutually independent Bernoulli random variables with probability mass functions
\[
\Pr[x_k = x] = \begin{cases} 
1/2 & x \in \{-2^{-k}, 2^{-k}\} \\
0 & \text{otherwise}
\end{cases}
\]
Let \( \sigma_k^2 \) denote the variance of \( x_k \), and \( S_n = \sqrt{\sum_{k=1}^{n} \sigma_k^2} \), and consider the sequence of random variables
\[
z_n = \frac{1}{S_n} \sum_{k=1}^{n} x_k
\]
Show that, as \( n \to \infty \), \( z_n \) converges in distribution to a uniformly distributed random variable.