ENEE 621 Estimation & Detection Theory
Exam 1 Answer Booklet
Wednesday, March 9, 2005
3:30–4:45pm

NAME: solutions

- Don't forget to put your name on all sheets.
- Only this answer booklet will be considered in the grading.
- Be sure to show all relevant work and reasoning.
- Neatness counts.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>11</td>
</tr>
<tr>
<td>2</td>
<td>17</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
</tr>
<tr>
<td>Total</td>
<td>50</td>
</tr>
</tbody>
</table>
Problem 1

(a) Show that \( \mathbf{u} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}) \):

First since \( \mathbf{w} \) is a Gaussian random vector and \( \mathbf{T} \) is a linear transformation, \( \mathbf{u} \) is also a Gaussian random vector.

\[
\mathbb{E}[\mathbf{u}] = \mathbb{E}[\mathbf{T}\mathbf{w}] = \mathbf{T}\mathbb{E}[\mathbf{w}] = \mathbf{T}\mathbf{0} = \mathbf{0}
\]

\[
\mathbf{A}_\mathbf{u} = \mathbb{E}[\mathbf{uu}^T] = \mathbb{E}[\mathbf{T}\mathbf{ww}^T\mathbf{T}^T] = \mathbf{T}\mathbb{E}[\mathbf{ww}^T]\mathbf{T}^T = \mathbf{T}\mathbf{A}_\mathbf{w}\mathbf{T}^T
\]

\[
= \lambda^{-1/2} \mathbf{V}^T \mathbf{W} \mathbf{V}^T \lambda^{-1/2} = \lambda^{-1/2} \mathbf{A}_\mathbf{w} \lambda^{-1/2} = \mathbf{I}
\]

So \( \mathbf{u} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}) \)

(b) Optimal rule description and associated \( \Pr(e) \):

\[
\hat{H}(\mathbf{y}) = \mathbf{H}_1 \quad \text{if} \quad \mathbf{x}^T \mathbf{A}_\mathbf{w}^{-1} \mathbf{y} > \theta
\]

\[
\hat{H}(\mathbf{y}) = \mathbf{H}_0 \quad \text{else}
\]

\[
\Pr(e) = \mathcal{Q} \left( \sqrt{\mathbf{x}^T \mathbf{A}_\mathbf{w}^{-1} \mathbf{x}} \right)
\]

Reasoning/Work to be looked at:

First consider \( \mathbf{z} = \mathbf{T}\mathbf{y} \). We note that since \( \mathbf{T} \) is invertible, \( \mathbf{z} \) carries the same info as \( \mathbf{y} \). [\( \mathbf{T} = \lambda^{-1/2} \mathbf{V}^{-T} \)]

So the optimal rule \( \hat{H}(\mathbf{y}) \) can be expressed as the cascade of \( \mathbf{y} \rightarrow [\mathbf{T}] \rightarrow \mathbf{z} \rightarrow [\hat{H}(\mathbf{z})] \rightarrow \hat{H}(\mathbf{y}) \)

where \( \hat{H}(\mathbf{z}) \) is the opt. rule for choosing \( \mathbf{H} \) given obs of \( \mathbf{z} \), we have given \( \mathbf{H} = \mathbf{H}_1 \); \( \mathbf{z} = \mathbf{x}_c' + \mathbf{u} \) where \( \mathbf{x}_c' = \mathbf{T}\mathbf{x}_c \) [known]

we have, \( \hat{H}(\mathbf{z}) = \mathbf{H}_1 \)

\[
\mathbf{L}(\mathbf{z}) \lesssim_{\mathbf{H}_0} \mathbf{Y} = \frac{1}{\sqrt{n}} \left[ \mathbf{I} - \mathbf{D} \right] = \mathbf{I}
\]

As we've shown in class the opt. rule is given by (since \( n = 1 \))

\[
\Delta_{\mathbf{x}_c'}^T \mathbf{z} \gtrsim \frac{1}{2} (\Delta_{\mathbf{x}_c'}^T)^T (\mathbf{x}_c' + \mathbf{y}_0')
\]
Problem 1  (continued)

(b) Reasoning/Work to be looked at: (continued)

\[
\Delta x' = x'_1 - x'_0 = 2T x = 2x', \quad w' = x'Tx', \quad x_1 + x_2 = \mathbb{E}
\]

we have

\[
\hat{H}'(z) = H_1, \quad (x'TT)z > \frac{\chi^2}{2}, \quad \emptyset
\]

\[
\hat{H}'(z) = H_0
\]

Using \( z = Ty \), we have

\[
x'TTz = x'TTT'y = x'TV\Lambda^{-\frac{1}{2}}\Lambda^{-\frac{1}{2}}VT'y = x'T\Lambda^{-1}y
\]

The proof of the optimal rule [from class] is

\[
P(y|x) = \mathcal{Q}\left( \frac{||\Delta x'||}{2\sigma} \right) \quad w' \quad \sigma = 1 \quad ||\Delta x'|| = 4x'TTTx = 4xT\Lambda^{-1}x
Problem 1 (continued)

(c) Optimal choice of $x$ and associated $Pr(e)$:

$$x = \sqrt{\frac{\mathbf{v}}{\mathbf{V}^2}} \cdot \mathbf{v} \cdot \mathbf{n}$$

$$Pr(e) = \mathcal{Q} \left( \frac{\sqrt{\mathbf{v}}}{\mathbf{V}^2} \right)$$

Reasoning/Work to be looked at:

Subject to $\mathbf{n}^T \mathbf{v}^2 \leq \varepsilon$ we wish to minimize $Pr(e)$, or, equivalently, maximize $\left[ x^T \mathbf{V}^{-1} x \right]$.

$$x^T \mathbf{V}^{-1} x = x^T \left[ \mathbf{V} \Lambda^{-1} \mathbf{V}^T \right]^{-1} x = x^T (\mathbf{V}^T \Lambda^{-1} \mathbf{V})^{-1} x = x^T \mathbf{V}^{-1} \mathbf{V}^T x$$

$$= \sum_{i=1}^{N} \frac{s_i^2}{\sigma_i^2}$$

Let $s = \mathbf{V}^T x$

But $\|s\|_2 = s^T s = x^T \mathbf{V} \mathbf{V}^T x = \|x\|_2^2$

So we need to maximize $\sum_{i=1}^{N} \frac{s_i^2}{\sigma_i^2}$ subject to $\sum_{i=1}^{N} s_i^2 \leq \varepsilon$

Clearly, the optimal choice is $s = [0 \ldots 0 \ 1]^T \sqrt{\mathbf{v}}$

yielding

$$\max_{s_i, \|s\|_2 \leq \sqrt{\mathbf{v}}} \sum_{i=1}^{N} \frac{s_i^2}{\sigma_i^2} = \frac{\varepsilon}{\varepsilon^2}$$

Thus, the optimal choice of $x$ is $x = \sqrt{s_{opt}} \cdot \mathbf{v} \cdot \mathbf{n}$.
Problem 2

(a) Specify all deterministic rules and their performance in the \((P_F, P_D)\) plane:

Any deterministic rule is a function \(\hat{H}(\cdot) : \{0,1\} \rightarrow \{H_0, H_1\}\). Clearly, there are \(2 \times 2 = 4\) possible distinct rules.

\(\hat{H}_1: \quad \hat{H}_1(y) = H_1 \quad \forall y, \quad (P_F, P_D) = (1, 1)\)

\(\hat{H}_2: \quad \hat{H}_2(y) = H_0 \quad \forall y, \quad (P_F, P_D) = (0, 0)\)

\(\hat{H}_3: \quad \hat{H}_3(y) = \begin{cases} H_0 & y = 0 \\ H_1 & y = 1 \end{cases}\)

\(P_F(\hat{H}_3) = P_Y[\hat{H}_3(y) = H_1 | H = H_0] = P_Y[y = 1 | H = H_0] = \frac{2}{3}\)

\(P_D(\hat{H}_3) = P_Y[\hat{H}_3(y) = 1 | H = H_1] = \frac{3}{4}\)

\(So \quad (P_F(\hat{H}_3), P_D(\hat{H}_3)) = \left(\frac{2}{3}, \frac{3}{4}\right)\)

\(\hat{H}_4: \quad \hat{H}_4(y) = \begin{cases} H_0 & y = 1 \\ H_1 & y = 0 \end{cases}\)

\(P_F(\hat{H}_4) = 1 - P_F(\hat{H}_3) = \frac{1}{3} \quad P_D(\hat{H}_4) = 1 - P_D(\hat{H}_3) = \frac{1}{4}\)

\(So \quad (P_F(\hat{H}_4), P_D(\hat{H}_4)) = \left(\frac{1}{3}, \frac{1}{4}\right)\)
Problem 2 (continued)

(b) $P_0$-maximizing deterministic rule as a function of $\alpha$:

<table>
<thead>
<tr>
<th>$\alpha$ range</th>
<th>opt rule</th>
<th>$P_0$ of opt rule</th>
<th>$P_F$ of opt rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset &lt; \alpha &lt; \frac{1}{3}$</td>
<td>$\hat{H}_2$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$\frac{1}{3} \leq \alpha &lt; \frac{2}{3}$</td>
<td>$\hat{H}_4$</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>$\frac{2}{3} \leq \alpha &lt; 1$</td>
<td>$\hat{H}_3$</td>
<td>$\frac{3}{4}$</td>
<td>$\frac{2}{3}$</td>
</tr>
<tr>
<td>$\alpha = 1$</td>
<td>$\hat{H}_1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

Reasoning/Work to be looked at:

For $0 \leq \alpha < \frac{1}{3}$, only one possible rule with $P_F < \alpha \Rightarrow \hat{H}_2$.

For $\frac{1}{3} \leq \alpha < \frac{2}{3}$, two possible rules: $\hat{H}_2$ and $\hat{H}_4$ (with $P_F < \alpha$).

Clearly, $P_D(\hat{H}_4) > P_D(\hat{H}_2)$.

For $\frac{2}{3} \leq \alpha < 1$, three possible rules: $\hat{H}_2$, $\hat{H}_4$, $\hat{H}_3$. Obviously,

$P_D(\hat{H}_3) > P_D(\hat{H}_4)$.

For $\alpha = 1$ (all rules are $P_F < 1$), pick $\hat{H}_1$. 


Problem 2  (continued)

(c) Description of deterministic rule that minimizes the worst-case Pr(e):

\[
\hat{H}(y) = \hat{H}_2(y) = \begin{cases} 
H_1, & y = 1 \\
H_0, & y = \emptyset 
\end{cases}
\]

Reasoning/Work to be looked at:

\[
P_\gamma(e) = P_D \cdot Pr(e | H=H_0) + P_1 \cdot Pr(e | H=H_1) = (1-P_D)P_F + P_1(1-P_D)
\]

\[
= P_F(\hat{H}) + P_1(1-P_D(\hat{H}) - P_F(\hat{H})) = P_0(P_1, \hat{H})
\]

\[
\hat{H}_1: \quad \max_{p_1 \in [0, 1]} P_0(p_1, \hat{H}_1) = \max_{p_1 \in [0, 1]} [1 - \frac{\bar{p}_1}{4}] = 1
\]

\[
\hat{H}_2: \quad \max_{p_1 \in [0, 1]} P_0(p_1, \hat{H}_2) = \max_{p_1 \in [0, 1]} p_1 = 1
\]

\[
\hat{H}_3: \quad \max_{p_1 \in [0, 1]} P_0(p_1, \hat{H}_3) = \max_{p_1 \in [0, 1]} \left[ \frac{2}{3} + p_1 \left(1 - \frac{3}{8} - \frac{3}{4}\right) \right]
\]

\[
= \max_{p_1 \in [0, 1]} \left[ \frac{2}{3} - p_1 \frac{5}{12} \right] = \frac{2}{3}
\]

\[
\hat{H}_4: \quad \max_{p_1 \in [0, 1]} P_0(p_1, \hat{H}_4) = \max_{p_1 \in [0, 1]} \left[ \frac{1}{3} + p_1 \left(1 - \frac{1}{3} - \frac{1}{4}\right) \right]
\]

\[
= \max_{p_1 \in [0, 1]} \left[ \frac{1}{3} + p_1 \frac{5}{12} \right] = \frac{1}{3} + \frac{5}{12} = \frac{9}{12} = \frac{3}{4}
\]
Problem 2 (continued)

(d) Description of rule that minimizes the worst-case \( \Pr(e) \):

\[
\Pr[\hat{H}(y) = H_1] = \begin{cases} 
\frac{5}{17} & y = 1 \\
0 & y = 0 
\end{cases}
\]

Reasoning/Work to be looked at:

The optimal rule is an LRT, or, equivalently, a randomized test between two "successive" deterministic LRTs.

The only deterministic LRT tests are \( \hat{H}_1, \hat{H}_2, \) and \( \hat{H}_3 \) since \( \hat{H}_3 \) can be phrased as

\[
\{ \hat{H}_3 = H_1 \} \quad \text{or} \quad \{ \hat{H}_3 = H_1 \}
\]

\[ \{ \hat{H}_3 \text{ is } \hat{H}_3 \text{ w.l. reversed decisions} \} \]

Thus we have two choices

\[
\begin{align*}
\hat{H}_a(y) &= \{ \hat{H}_3(y) \} \text{ w.l. prob } q \\
\hat{H}_b(y) &= \{ \hat{H}_1(y) \} \text{ w.l. prob } (1-q)
\end{align*}
\]

\[
P_e(P_1, \hat{H}_a(y)) = q P_e(P_1, \hat{H}_3) + (1-q) P_e(P_1, \hat{H}_3)
\]

\[
= q P_1 + (1-q) \left[ \frac{2}{3} - P_1 \frac{5}{12} \right] = \frac{2}{3} - \frac{5}{12} P_1 + P_1 \left[ q + \frac{5}{12} \right]
\]

\[
\frac{3}{2} P_1 P_e(P_1, \hat{H}_a(y)) = 0 \Rightarrow q + \frac{5}{12} q = 1 \Rightarrow q = \frac{12}{17} \text{ opt rule is } \hat{H}_a(y)
\]

Note that:

\[
P_e(P_1, \hat{H}_b(y)) = q \left[ \frac{2}{3} - P_1 \frac{5}{12} \right] + (1-q)(1-P_1) = 1 - \frac{1}{3} P_1 - P_1 \left[ q + \frac{5}{12} \right]
\]

\[
\frac{3}{2} P_1 P_e(P_1, \hat{H}_b(y)) = 0 \Rightarrow 1 - \frac{7}{12} q = 0 \Rightarrow q = \frac{12}{7} > 1
\]

\[
\text{Hence it is not possible to get } 0 \text{-slope w.l. } \hat{H}_b(y). \text{ [as expected]}
\]
Problem 3

(a) Description of decision rule \( H_0 \) (or, equivalently, \( \eta_1, \eta_2, \) and \( q \)) that maximizes \( P_D \) subject to \( P_F \leq \frac{3}{8} \): 

\[
\eta_1 = \frac{3}{2}, \quad \eta_2 = 2, \quad q = \frac{1}{2}
\]

Reasoning/Work to be looked at:

\[
L_1(y) \quad \hat{H}_1 \Rightarrow \begin{cases} \eta_1 = \frac{1}{2} \Rightarrow (P_F, P_D) = (1, 1) \\ \eta_1 = \frac{3}{2} \Rightarrow (P_F, P_D) = \left( \frac{1}{2}, \frac{3}{4} \right) \\ \eta_1 > \frac{3}{2} \Rightarrow (P_F, P_D) = (0, 0) \end{cases}
\] shown w/ "x" marks on \((P_F, P_D)\) plane

\[
L_2(y) \quad \hat{H}_1 \Rightarrow \begin{cases} \eta_2 = \frac{1}{2} \Rightarrow (P_F, P_D) = (1, 1) \\ \eta_2 = 2 \Rightarrow (P_F, P_D) = \left( \frac{1}{4}, \frac{1}{2} \right) \\ \eta_2 > 2 \Rightarrow (P_F, P_D) = (0, 0) \end{cases}
\] shown w/ "o" marks on \((P_F, P_D)\) plane

\[
P_F(\hat{H}_1) = q \cdot P_F(\hat{H}_1) + (1-q) \cdot P_F(\hat{H}_2)
\]

\[
P_D(\hat{H}_1) = q \cdot P_D(\hat{H}_1) + (1-q) \cdot P_D(\hat{H}_2)
\]

Clearly, to max \( P_D \) subject to \( P_F \leq \frac{3}{8} \) we need to randomize between tests w/ \( \eta_1 = \frac{3}{2} \) 

\[
(\eta_1, \eta_2) = \left( \frac{3}{2}, \frac{3}{4} \right) \quad (P_F, P_D) = \left( \frac{1}{2}, \frac{1}{2} \right)
\]
Problem 3 (continued)

(a) Reasoning/Work to be looked at: (continued)

\[
\text{Since } \frac{\frac{3}{8} - \frac{1}{4}}{\frac{1}{2} - \frac{1}{4}} = \frac{1}{2} \implies q = \frac{1}{2}
\]

the associated \( P_D \) is

\[
P_D(\hat{H}_c) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{3}{4} = \frac{5}{8}
\]

\[
1 - q \cdot P_D(\hat{H}_{(e)}) \cdot P_D(\hat{H}_{(n)})
\]
Problem 3 (continued)

(b) Sketch as much of the ROC of decision rules based on \( y \), as you can.

![ROC Graph]

Reasoning/Work to be looked at:

\[
L(y) = \frac{p_{Y_i, Y_2 \mid H_i}(y_i, y_2 \mid H_i)}{p_{Y_i, Y_2 \mid H_0}(y_i, y_2 \mid H_0)} = \frac{p_{Y_1 \mid H_i}(y_1 \mid H_i)p_{Y_2 \mid H_i}(y_2 \mid H_i)}{p_{Y_1 \mid H_0}(y_1 \mid H_0)p_{Y_2 \mid H_0}(y_2 \mid H_0)} = L_1(y_1)L_2(y_2)
\]

Clearly, since \( L_1 \in \{\frac{1}{3}, \frac{2}{3}, 2\} \), \( L_2 \in \{\frac{3}{8}, \frac{5}{8}\} \), \( L \) takes values in \( \{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}\} \).

\[
P_{L \mid H} \left[ \frac{1}{3} \mid H_0 \right] = p_{L \mid H} \left[ \frac{1}{3} \mid H \right]p_{L \mid H} \left[ \frac{2}{3} \mid H \right] = \left\{ \begin{array}{ll} \frac{1}{8} & c = 1 \\ \frac{3}{8} & c = 0 \end{array} \right.
\]

Similarly,

\[
P_{L \mid H} \left[ \frac{3}{8} \mid H_0 \right] = p_{L \mid H} \left[ \frac{3}{8} \mid H \right]p_{L \mid H} \left[ \frac{5}{8} \mid H \right] = \left\{ \begin{array}{ll} \frac{7}{8} & c = 1 \\ \frac{5}{8} & c = 0 \end{array} \right.
\]

\[
P_{L \mid H} \left[ \frac{5}{8} \mid H_0 \right] = 1 - p_{L \mid H} \left[ \frac{1}{3} \mid H \right] - p_{L \mid H} \left[ \frac{3}{8} \mid H \right] = \frac{1}{8} \quad \text{for} \quad c = 0, 1
\]

Hence:

\[
\frac{1}{8} \quad \frac{3}{8} \quad 1
\]

\[
\frac{1}{3} \quad \frac{2}{3} \quad 3
\]
Problem 3 (continued)

(b) Reasoning/Work to be looked at: (continued)

- To get ROC, first get detem. LRTs

\[
\begin{array}{c|c}
\text{ } & (P_F, P_D) \\
\hline
\frac{n}{\theta} & (1, 1) \\
\frac{5}{8} & (\frac{5}{8}, \frac{7}{8}) \\
\frac{1}{\theta} & (\frac{1}{8}, \frac{3}{8}) \\
\frac{3}{8} & (0, 0) \\
\end{array}
\]

- Connect consecutive \( P_F \) pts with linear segment \( \rightarrow \) OC of LRT

- Get the \( P_F, P_D \) of inverted LRTs

\[
\begin{array}{c|c}
\text{ } & (P_F, P_D) \\
\hline
\frac{n}{\theta} & (1 - \frac{5}{8}, 1 - \frac{3}{8}) = (\frac{3}{8}, \frac{5}{8}) \\
\frac{1}{\theta} & (1 - \frac{1}{8}, 1 - \frac{3}{8}) = (\frac{7}{8}, \frac{5}{8}) \\
\end{array}
\]

- Connect consecutive \( P_F \) pts (inv. LRTs) with linear segment gives \( \text{OC of inverted LRTs} \)

ROC = all pts in region bounded by \( \text{OC of LRT} \) & \( \text{OC of LRTs} \) w/ inverted decisions.
Problem 4

(a) Does a UMP test exist for testing $H_0 : x = 1$ vs. $H_1 : x > 1$, subject to $\alpha = 0.1$?

Circle one:  yes  no  maybe

Reasoning/Work to be looked at:

Consider the opt. $\alpha$-size test for testing $x = 1$ vs. $x = x_1$ for some fixed $x_1 > 1$. The optimal test is an LRT

$$L(y) = \frac{P_y(y | x_1)}{P_y(y | 1)}$$

$$\frac{\hat{y}(y) = H_1}{\hat{y}(y) = H_0}$$

where $\eta$ and $\tilde{z}$-randomization when $L = \eta$ are chosen so that $P_Y[\hat{H}(y) = H_1; x = 1] = \alpha$

$$(*) \iff x \leq x_1 \frac{\ln \left( \frac{x_1}{\eta} \right)}{1 - x_1}$$

Since $P_Y[L = \eta; x] = P_Y[Y = k(x)] = \alpha$ for $x$, randomization can be ignored (applies on a set of $y$'s of measure $B$).

$$P_F(H_1) = \alpha \iff P_Y[\hat{H}(y) = H_1; x = 1] = P_Y[Y = k; x = 1] = \alpha \iff$$

$$\iff \int_0^\infty e^{-y} dy = \alpha \iff 1 - e^{-k} = \alpha \iff$$

$$\iff \ln \left( \frac{1}{1 - \alpha} \right) = k = \ln \left( \frac{10}{3} \right)$$

Since $\Gamma_1 = \{y ; y < \ln \left( \frac{10}{3} \right) \}$,

$\hat{H}(y)$ in $(*)$ with $k = \ln (10/3)$ is the UMP test, subject to $\alpha = 0.1$.
Problem 4 (continued)

(b) Does a UMP test exist for testing $H_0 : x = 1$ vs. $H_1 : x \in \{0.5, 2\}$, subject to $\alpha = 0.1$?
Circle one: yes  \(\boxed{\text{no}}\)  maybe

Reasoning/Work to be looked at: (continued)

From part (a), for testing $H_0 : x = 1$ vs. $H_1 : x = 2 > 1$, the MP test is given by

\[
F(y) = \begin{cases} 
H_0 & \text{if } y \leq 2 \\
H_1 & \text{if } y > 2
\end{cases}
\]

\[
Y \geq k' \quad \Rightarrow \quad \ln\left(\frac{\frac{1}{2}}{1 - \frac{1}{2}}\right) = \ln(10)
\]

\[
F(y) = H_1
\]

The MP test for testing $H_0 : x = 1$ vs. $H_1 : x = \frac{1}{2}$ is given by \(\boxed{\text{in part (a)}}, \text{ for } x_1 = \frac{1}{2}. \text{ Since } x_1 < \frac{1}{2}, \text{ we get}

\[
Y \geq k' \quad \Rightarrow \quad \ln\left(\frac{n}{x_1}\right) / \left(1 - x_1\right) \quad x_1 = \frac{1}{2}
\]

where $k$ is chosen s.t. $P_F(H_1') = \alpha \iff$

\[
P_x [\hat{H}(y) = H_1 \text{ i.f.}] = \alpha \iff P \left[ y > k' \right] = \int_{k'}^{\infty} e^{-y} dy = \alpha
\]

\[
e^{-k'} = \alpha \iff \ln(\frac{1}{\alpha}) = \ln(10)
\]

\[
F_1' (x) = (\ln(10)) \Rightarrow \text{ since } F_1' \neq F_1 \text{ (in fact, } F_1' \cap F_1 = \emptyset) \text{ the two MP tests are different, hence no UMP exists.}