1. The sample space Ω being uncountable, it is not possible for a set $E$ to be countable and for its complement $E^c$ to be uncountable

1.a. (i) Obviously $\emptyset$ and $\Omega$ belong to $\mathcal{F}$; (ii) Complementarity: Pick $E$ in $\mathcal{F}$. If $E$ is countable, then the complement of $E^c$, namely $(E^c)^c = E$, is also countable so $E^c \in \mathcal{F}$. If $E^c$ is countable, then $E^c \in \mathcal{F}$ automatically; (iii) Closed under union: Pick $E$ and $F$ in $\mathcal{F}$. Two cases are possible when considering $E \cup F$: If both $E$ and $F$ are countable, then their union $E \cup F$ is also countable and is therefore an element of $\mathcal{F}$. If one of $E$ and $F$ has a countable complement, note that $(E \cup F)^c = E^c \cap F^c$ is countable (since one of $E^c$ or $F^c$ is countable), hence $E \cup F \in \mathcal{F}$.

1.b. (iv) Closed under countable union: Let $\{E_n, n = 1, 2, \ldots \}$ be a countable collection of events in $\mathcal{F}$. If $E_n$ is a countable set for each $n = 1, 2, \ldots$, then the union $\bigcup_{n=1}^{\infty} E_n$ is also a countable set, hence is an element of $\mathcal{F}$. If one of the elements in the collection, say $E_m$ for some $m = 1, 2, \ldots$, has a countable complement, then $(\bigcup_{n=1}^{\infty} E_n)^c = \bigcap_{n=1}^{\infty} E_n^c \subseteq E_m^c$ is a countable set and $\bigcup_{n=1}^{\infty} E_n$ is therefore an element of $\mathcal{F}$.

1.c. We begin with a simple fact:

**Fact 0.1** Consider the events $E$ and $F$ in $\mathcal{F}$. If $E^c$ and $F^c$ are both countable, then they cannot be disjoint.

**Proof.** If $E \cap F = \emptyset$, then $(E \cap F)^c = E^c \cup F^c = \Omega$. But $E^c$ and $F^c$ being countable implies $E^c \cup F^c$ being countable, and this yields a contradiction since $E^c \cup F^c = \Omega$ with $\Omega$ assumed uncountable!

The conditions $\mathbb{P}[\emptyset] = 0$ and $\mathbb{P}[\Omega] = 1$ are immediate by definition. Pick a countable collection of events $\{F_i, i \in I\}$ in $\mathcal{F}$ such that

$$F_i \cap F_j = \emptyset, \quad i \neq j, \quad i, j \in I.$$
There are two cases:

(i) The sets \( \{ F_i, \ i \in I \} \) are all countable: Then, \( F = \bigcup_{i \in I} F_i \) is countable, and noting that \( P[F_i] = 0 \) for each \( i \) in \( I \) and \( P[F] = 0 \), it is plain that \( P[F] = \sum_{i \in I} P[F_i] \).

(ii) Some of the \( \{ F_i, \ i \in I \} \) have a countable complement: By Fact 0.2, there can be only one such set under the condition that the events are pairwise disjoint. Thus, there exists a single index \( i^* \) in \( I \) such that \( F_{i^*}^c \) is countable and \( F_i \) is countable for all \( i \neq i^* \) in \( I \). As before, we see that \( F = \bigcup_{i \in I} F_i \) has a countable complement, so that \( P[F] = 1 \), while \( P[F_{i^*}] = 1 \) and \( P[F_i] = 0 \) for all \( i \neq i^* \) in \( I \). Obviously, \( P[F] = \sum_{i \in I} P[F_i] \) since

\[
\sum_{i \in I} P[F_i] = P[F_{i^*}] + \sum_{i \in I: i \neq i^*} P[F_i] = 1 + \sum_{i \in I: i \neq i^*} 0.
\]

2.

2.a. For experiment \( \mathcal{E}_1 \) a simple and natural model that will help you answer the question is as follows: Take \( \Omega = \{1, \ldots, P\}^\mathbb{N}_0 \), the collection of all infinite sequences drawn from the alphabet \( \{1, \ldots, P\} \) – Thus, an element \( \omega \) of \( \Omega \) is of the form \( \omega = (k_1, k_2, \ldots, k_n, \ldots) \) with \( k_1, k_2, \ldots, k_n, \ldots \) elements of \( \mathcal{P} \). Here, as for the experiment of throwing a coin infinitely often under identical and independent conditions, use \( F = \sigma(\mathcal{G}) \) where

\[
\mathcal{G} \equiv \{ A_n(k_n), \ k_n \in \mathcal{P} \}
\]

with

\[
A_n(k_n) = \{ \omega = (\omega_1, \omega_2, \ldots) \in \Omega : \omega_n = k_n \}, \quad k_n \in \mathcal{P}.
\]

The description of \( \mathcal{E}_1 \) suggests that the defining conditions for \( P_1 \) should be that

\[
P[A_n(k_n)] = \frac{1}{P}, \quad k_n \in \mathcal{P}, \quad n = 1, 2, \ldots \tag{1.1}
\]

and that for each \( n = 1, 2, \ldots \), the events \( A_n(k_n), \ldots, A_n(k_n) \) are mutually independent for arbitrary \( k_1, \ldots, k_n \) in \( \mathcal{P} \), namely

\[
P[\cap_{\ell=1}^n A_\ell(k_\ell)] = \prod_{\ell=1}^n P[A_\ell(k_\ell)] = \frac{1}{P^n}, \quad k_1 \in \mathcal{P}, \ldots, k_n \in \mathcal{P} \quad n = 1, 2, \ldots \tag{1.2}
\]

By Caratheodory’s Extension Theorem there exists a unique probability measure \( P \) on \( \mathcal{F} = \sigma(\mathcal{G}) \) that satisfies (1.1)-(1.2).

This model takes the position that one continues to draw the balls even after \( K \) distinct balls have been drawn. Another model could have been obtained by taking the set \( \Omega_1 \) to be the collection of all finite length sequences \( (\omega_1, \ldots, \omega_n) \) drawn from \( \mathcal{P} \) satisfying the conditions

\[
\omega_i \in \mathcal{P}, \quad i = 1, \ldots, n
\]

and

\[
|\{\omega_1, \ldots, \omega_{n-1}\}| = K - 1 \quad \text{and} \quad \omega_n \notin \{\omega_1, \ldots, \omega_{n-1}\}.
\]
You might wonder why not use this model (as some of the students in the class have done) – after all it sticks very closely to the description of $E_1$. It is because calculations in that model are more difficult (with very few of the adopters correctly completing them).

With the first model suggested above, we note that $P_1[\Sigma_1 = S]$ cannot depend on the set $S$. Since there are $\binom{P}{K}$ such sets of size $K$ (as order is not important), we immediately conclude that $P_1[\Sigma_1 = S] = \binom{P}{K}^{-1}$. Of course it is possible to calculate this quantity by brute force – Try it!

2.b. For $E_2$ take $\Omega$ to be the collection of all ordered subsets of size $K$ made up of $K$ distinct elements drawn from the set $P$ – there are $K!\binom{P}{K} = P(P-1)\ldots(P-K+1)$ such ordered subsets! Since $\Omega_2$ is finite we take $F_2 = P(\Omega_2)$. To define $P_2$ we need only define it on singletons. To do so, pick $\alpha = (\alpha_1, \ldots, \alpha_K)$ in $\Omega_2$, and define the events

$$A_k(\alpha_k) = \{\omega \in \Omega_2 : \omega_k = \alpha_k\}, \quad k = 1, 2, \ldots, K.$$  

In other words, $A_k(\alpha_k)$ is the event where the $k^{th}$ draw is $\alpha_k$. The description of $E_2$ leads to

$$P_2[A_1(\alpha_1)] = \frac{1}{P}$$

(reflecting the fact that the first ball is drawn uniformly from $P$) and

$$P_2[A_k(\alpha_k)|A_1(\alpha_1)\cap \ldots \cap A_{k-1}(\alpha_{k-1})] = \frac{1}{P - (k-1)}, \quad k = 2, \ldots, K$$

(reflecting the fact that once the distinct balls $\alpha_1, \ldots, \alpha_{k-1}$ have been drawn, there remains $P - (k-1)$ balls, and the $k^{th}$ is drawn uniformly at random from these remaining balls). To conclude, since

$$\{\alpha\} = \cap_{k=1}^K A_k(\alpha_k)$$

we conclude that

$$P_2[\{\alpha\}] = P_2[\cap_{k=1}^K A_k(\alpha_k)] = \prod_{k=2}^K P_2[A_k(\alpha_k)|A_{k-1}(\alpha_{k-1})] \cdot P_2[A_1(\alpha_1)] = \prod_{k=1}^K \frac{1}{P - (K-k)} = \frac{(P-K)!}{P!}.$$  

This final expression is a consequence of the assumptions of the model (implied by the description of $E_2$) and not an assumption as many seem to have implied in their answers. Finally,

$$P_2[\Sigma_2 = S] = \sum_{\omega \in \Omega_2 : \text{Set}[\omega] = S} P_2[\{\omega\}] = \frac{(P-K)!}{P!} \cdot K! = \binom{P}{K}^{-1}$$

where $\text{Set}[\omega]$ denotes the unordered set associated with $\omega$.

2.c. These probabilities are identical as the calculations have revealed. But this could have been guessed in advance through the following argument: In each experiment,
without constructing the appropriate probability model, it is plain that the mechanics
of the experiment do not bias the outcome as far as the \( K \) elements finally collected
are concerned. Thus, in each experiment, any set \( S \) is ascribed the same likelihood of
occurrence. Since there are \( \binom{P}{K} \) such sets, in both cases the answer will be \( \binom{P}{K}^{-1}! \)

3. 

3.a. A natural model for this situation is the following: Imagine that the hats and
their owners have been labelled 1, \ldots, \( n \) as they are handed out to the coat check: The
hat labelled \( k \) is that of the \( k^{th} \) person (who receives label \( k \)). Thus, at the end of the
evening, when the hats are returned, let \( \omega(k) \) be the label of the person that receives the
hat labelled \( k \). Obviously, \( \omega = (\omega(1), \ldots, \omega(n)) \) is a permutation of \( \{1, \ldots, n\} \), and it is
natural to take \( \Omega \) to be the collection of all permutations of \( \{1, \ldots, n\} \). To completely
specify the model, set \( F = \mathcal{P}(\Omega) \) and use the uniform probability assignment for \( \mathcal{P} \), i.e.,

\[
\mathbb{P}\left[\{\omega\}\right] = \frac{1}{|\Omega|} = \frac{1}{n!}, \quad \omega \in \Omega.
\]

3.b. For each \( k = 1, \ldots, n \), we have \( X_k = 1_E[k] \) where \( E[k] \equiv \{\omega \in \Omega : \omega(k) = k\} \). Note
that \( |E[k]| = (n-1)! \) so that

\[
\mathbb{P}[X_k = 1] = \frac{|E[k]|}{n!} = \frac{1}{n}.
\]

and

\[
\mathbb{E}[S_n] = \sum_{k=1}^{n} \mathbb{E}[X_k] = 1.
\]

3.c. For distinct \( k, \ell = 1, \ldots, n \), we have \( |E[k] \cap E[\ell]| = (n-2)! \) so that

\[
\mathbb{P}[X_k = 1, X_\ell = 1] = \frac{|E[k] \cap E[\ell]|}{n!} = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}.
\]

Obviously

\[
\mathbb{P}[X_k = 1] \cdot \mathbb{P}[X_\ell = 1] = \frac{1}{n^2} \neq \frac{1}{n(n-1)} = \mathbb{P}[X_k = 1, X_\ell = 1]
\]

and the rvs \( X_k \) and \( X_\ell \) are not independent. We also get

\[
\text{Cov}[X_k, X_\ell] = \mathbb{P}[X_k = 1, X_\ell = 1] - \mathbb{P}[X_k = 1] \cdot \mathbb{P}[X_\ell = 1]
= \frac{1}{n(n-1)} - \frac{1}{n^2}
= \frac{n - (n-1)}{n^2(n-1)} = \frac{1}{n^2(n-1)} \neq 0 \quad (1.4)
\]

and the rvs \( X_k \) and \( X_\ell \) are not uncorrelated.
3.d. By the standard expression for the variance of a sum, we get
\[ \text{Var} [S_n] = \sum_{k=1}^{n} \text{Var} [X_k] + \sum_{k=1}^{n} \sum_{\ell \neq k} \text{Cov} [X_k, X_\ell] \]
\[ = n \left( 1 - \frac{1}{n} \right) \frac{1}{n} + n(n-1) \frac{1}{n^2(n-1)} \]
\[ = 1. \] 

(1.5)

3.e. Note that
\[ \mathbb{P} [S_n \geq 11] = \mathbb{P} [S_n - 1 \geq 10] \]
\[ \leq \mathbb{P} [\|S_n - \mathbb{E} [S_n]\| \geq 10] \quad [\text{Recall that } \mathbb{E} [S_n] = 1] \]
\[ \leq \mathbb{E} [\|S_n - \mathbb{E} [S_n]\|^2] \quad [\text{By Byenaimé-Tchebychev}] \]
\[ = \frac{\text{Var} [S_n]}{100} = \frac{1}{100} = 0.01. \] 

(1.6)

4.

4.a. Note that the probability distributions of \( X \) and \( Y \) coincide under the condition \( X = Y \) a.s.: Indeed, we have \( F_X = F_Y \) since
\[ \mathbb{P} [X \leq x] = \mathbb{P} [X = Y, X \leq x] + \mathbb{P} [X \neq Y, X \leq x] \]
\[ = \mathbb{P} [X = Y, Y \leq x] \]
\[ = \mathbb{P} [X = Y, Y \leq x] + \mathbb{P} [X \neq Y, Y \leq x] \]
\[ = \mathbb{P} [Y \leq x], \quad x \in \mathbb{R}. \] 

(1.7)

Let \( F_{\text{Common}} \) denote this common probability distribution function. Next we have
\[ F_{X,Y}(x,y) = \mathbb{P} [X \leq x, Y \leq y] \]
\[ = \mathbb{P} [X = Y, X \leq x, Y \leq y] + \mathbb{P} [X \neq Y, X \leq x, Y \leq y] \]
\[ = \mathbb{P} [X = Y, X \leq x, Y \leq y] \]
\[ = \mathbb{P} [X = Y, X \leq x, X \leq y] \]
\[ = \mathbb{P} [X = Y, X \leq \min(x,y)] \]
\[ = \mathbb{P} [X = Y, X \leq \min(x,y)] + \mathbb{P} [X \neq Y, X \leq \min(x,y)] \]
\[ = \mathbb{P} [X \leq \min(x,y)] \]
\[ = F_{\text{Common}} (\min(x,y)). \] 

(1.8)

4.b. Assume that the function \( F : \mathbb{R}^2 \to [0,1] \) given by \( F(x,y) = \) for all \( x \) and \( y \) in \( \mathbb{R} \), is indeed the joint probability distribution of a pair of rvs \( U \) and \( V \) defined on some probability triple \((\Omega, \mathcal{F}, \mathbb{P})\). Thus, under the enforced assumptions we have
\[ F(x,y) = \mathbb{P} [U \leq x, V \leq y] = K(\min(x,y)), \quad x, y \in \mathbb{R} \]
By letting \( y \) (resp. \( x \)) go to infinity with \( x \) (\( y \)) fixed we get
\[
\mathbb{P}[U \leq x] = K(x), \quad x \in \mathbb{R}
\]
and
\[
\mathbb{P}[V \leq y] = K(y), \quad y \in \mathbb{R}
\]
Hence, both \( U \) and \( V \) have the same probability distribution function \( K : \mathbb{R} \to [0, 1] \) under \( \mathbb{P} \). More precisely, the function \( K : \mathbb{R} \to [0, 1] \) is a probability distribution function!

Recall that the function \( F : \mathbb{R}^2 \to [0, 1] \) is a probability distribution function if and only if (i) right-continuous in each component (ii) has a left-limit in each component (iii) \( \lim_{\min(x,y) \to -\infty} F(x, y) = 0 \) and \( \lim_{\min(x,y) \to \infty} F(x, y) = 1 \) and (iv) the conditions
\[
F(b, \beta) - F(b, \alpha) - (F(a, \beta) - F(a, \alpha)) \geq 0, \quad \alpha \leq b \leq \beta
\]
hold where (iv) amounts to the condition
\[
\mathbb{P}[a < U \leq b, \alpha < V \leq \beta] \geq 0, \quad \alpha \leq b \leq \beta.
\]

Conditions (i)-(iii) are inherited from the probability distribution function \( K : \mathbb{R} \to [0, 1] \).

The last condition (iv)
\[
K(\min(b, \beta)) - K(\min(b, \alpha)) \geq K(\min(a, \beta)) - K(\min(a, \alpha)), \quad \alpha \leq b \leq \beta
\]
is equivalent to the monotonicity of the probability distribution function \( K : \mathbb{R} \to [0, 1] \).

4.c. Assume that the function \( F : \mathbb{R}^2 \to [0, 1] \) given by \( F(x, y) = K(\min(x, y)) \) for all \( x \) and \( y \) in \( \mathbb{R} \), is indeed the joint probability distribution of a pair of rvs \( U \) and \( V \) defined on some probability triple \((\Omega, \mathcal{F}, \mathbb{P})\). We have seen earlier that \( \mathbb{P}[U \leq x] = K(x) \) for all \( x \) in \( \mathbb{R} \) and \( \mathbb{P}[V \leq y] = K(y) \) for all \( y \) in \( \mathbb{R} \). Therefore, the rvs \( U \) and \( V \) are independent if and only if
\[
\mathbb{P}[U \leq x, V \leq y] = \mathbb{P}[U \leq x] \cdot \mathbb{P}[V \leq y] = K(x)K(y), \quad x, y \in \mathbb{R}
\]
while
\[
\mathbb{P}[U \leq x, V \leq y] = K(\min(x, y)), \quad x, y \in \mathbb{R}.
\]
Therefore, we have independence if and only if
\[
K(\min(x, y)) = K(x)K(y), \quad x, y \in \mathbb{R}
\]
With \( x = y \), we get
\[
K(x) = K(x)^2, \quad x \in \mathbb{R}
\]
so that either \( K(x) = 0 \) or \( K(x) = 1 \). But, \( K : \mathbb{R} \to [0, 1] \) being a probability distribution function (by Part b), we conclude that \( K(x) = 0 \) if \( x < x^* \) and \( K(x) = 1 \) if \( x^* \leq x \) for some finite \( x^* \).
Hence, if the rvs $U$ and $V$ are independent, they must be degenerate in the sense that $U = x^*$ and $V = x^*$ a.s. for some finite $x^*$. Conversely, if $U = x^*$ and $V = x^*$ a.s. for some finite $x^*$, then the rvs $U$ and $V$ are automatically independent and they satisfy the requirement $P[U \leq x, V \leq y] = K(\min(x, y))$ for all $x$ and $y$ in $\mathbb{R}$.

4.d. Assume that the function $F : \mathbb{R}^2 \to [0, 1]$ given by $F(x, y) = \mathbb{P}[U \leq x, V \leq y] = K(\min(x, y))$, $x, y \in \mathbb{R}$.

By letting $y$ (resp. $x$) go to infinity while keeping $x$ (resp. $y$) fixed we get

$$\mathbb{P}[U \leq x] = K(x), \quad x \in \mathbb{R}$$

and

$$\mathbb{P}[V \leq y] = K(y), \quad y \in \mathbb{R}$$

Also, with $x = y = t$, we get

$$\mathbb{P}[\max(U, V) \leq t] = \mathbb{P}[U \leq t, V \leq t] = K(t), \quad t \in \mathbb{R}$$

so the three rvs $U$, $V$ and $\max(U, V)$ have the same probability distribution $K : \mathbb{R} \to [0, 1]$. Thus, for each $t$ in $\mathbb{R}$, we have

$$0 = \mathbb{P}[U \leq t] - \mathbb{P}[\max(U, V) \leq t]$$

$$= \mathbb{P}[U \leq t] - \mathbb{P}[U \leq t, V \leq t]$$

$$= \mathbb{P}[U \leq t < V], \quad t \in \mathbb{R}. \quad (1.10)$$

Consider $\omega$ in $\Omega$ such that $U(\omega) < V(\omega)$. It necessarily exists $t$ in $\mathbb{Q}$ such that $U(\omega) < t < V(\omega)$, whence

$$[U < V] = \bigcup_{t \in \mathbb{Q}} [U < t < V]$$

and a union bound argument shows that

$$\mathbb{P}[U < V] = \mathbb{P}[\bigcup_{t \in \mathbb{Q}} [U < t < V]]$$

$$\leq \sum_{t \in \mathbb{Q}} \mathbb{P}[U < t < V] = 0 \quad (1.11)$$

In other words, $\mathbb{P}[U < V] = 0$. By symmetry, $\mathbb{P}[V < U] = 0$, whence $\mathbb{P}[U \neq V] = 0$.

If $\mathbb{E}[U]$ exists and is finite (a condition on $K$), then the following argument could be used: Obviously, $U \leq \max(U, V)$ and $V \leq \max(U, V)$. Assume that so does $\mathbb{E}[U]$ and $\mathbb{E}[\max(U, V)]$, then $\mathbb{E}[\max(U, V) - U] = \mathbb{E}[\max(U, V)] - \mathbb{E}[U] = \mathbb{E}[U] - \mathbb{E}[U] = 0$ with $\max(U, V) - U \geq 0$. It follows that $\max(U, V) = U$ a.s. with a similar argument for $\max(U, V) = V$ a.s. We conclude that $U = V$ a.s.

5. This problem relies on the following basic facts:
Fact 0.2 The expectation of any bounded rv \( \eta : \Omega \to \mathbb{R} \) (i.e., \( \mathbb{P}[|\eta| \leq B] = 1 \) for some \( B > 0 \)) always exists and is finite with \( |E[\eta]| \leq B \).

Fact 0.3 If the rv \( \eta : \Omega \to \mathbb{R}^p \) is a symmetric rv (i.e., the rvs \( \eta \) and \( -\eta \) have the same probability distribution), then \( g(\eta) \) and \( g(-\eta) \) have the same probability distribution for any Borel mapping \( g : \mathbb{R}^p \to \mathbb{R}^q \).

Proof. For any mapping \( g : \mathbb{R} \to \mathbb{R} \), we have
\[
\mathbb{P}[g(-\xi) \in B] = \mathbb{P}[-\xi \in g^{-1}(B)] = \mathbb{P}[\xi \in g^{-1}(B)] \quad \text{[By symmetry}} \xi =_{st} -\xi] = \mathbb{P}[g(\xi) \in B]. \tag{1.12}
\]
In other words, the rvs \( g(\xi) \) and \( g(-\xi) \) have identical distribution.

Fact 0.4 If the rv \( \eta : \Omega \to \mathbb{R} \) is a symmetric rv, then it is always the case that \( E[\eta^+] = E[\eta^-] \). Only when the common value of these expectations is \textbf{finite} is it the case that \( E[\eta] = E[\eta^+] - E[\eta^-] = 0! \)

Proof. If the rv \( \eta : \Omega \to \mathbb{R} \) is a symmetric rv, then \( \xi^+ \) and \( \xi^- \) have the same distribution by Fact 0.3. Hence, \( E[\eta^+] = E[\eta^-] \), these expectations existing because the rvs are non-negative.

5.a. With \( X \equiv \sin (\xi) \), we have \( |X| \leq 1 \), and \( E[X] \) always exists and is finite. With
\[
Y \equiv \frac{\xi}{1+\xi^2},
\]
it is plain that we also have \( |Y| \leq 1 \) – Remember that \( |u| \leq 1+u^2 \) for each \( u \) in \( \mathbb{R} \). Thus, \( E[Y] \) always exists and is finite. With
\[
Z \equiv \xi \cdot \cos (\xi),
\]
it holds that \( |\cos (\xi)| \leq 1 \). However, unless additional conditions are imposed on \( \xi \), it is possible that \( E[Z] \) may not exist. It is easy to check that the rv \( Z \) is a symmetric rv when \( \xi \) is a symmetric rv, and \( E[Z] \) will not exist if and only if \( E[Z^+] = E[Z^-] = \infty \) (see below).

5.b. Obviously \( \sin (-\xi) = -\sin (\xi) \) while \( \sin (-\xi) =_{st} \sin (\xi) \). The expectations exist by boundedness, so \( E[-\sin (\xi)] = E[\sin (\xi)] \), whence \( E[X] = 0 \).

5.c. In a similar way, \( \frac{-\xi}{1+(-\xi)^2} \) while \( \frac{-\xi}{1+(\xi)^2} =_{st} \frac{\xi}{1+|\xi|^2} \). The expectations exist by boundedness, so \( E\left[\frac{-\xi}{1+\xi^2}\right] = E\left[\frac{\xi}{1+\xi^2}\right] \), whence \( E\left[\frac{\xi}{1+\xi^2}\right] = 0 \).
5.d. Consider a discrete rv $\xi$ with support $S = 2\pi \mathbb{Z}$ and pmf given by

$$p_{\xi}(2\pi z) = \frac{C}{1 + |z|^2}, \quad z \in \mathbb{Z}$$

with $C > 0$ determined by

$$C \left(1 + 2 \sum_{z=1}^{\infty} \frac{1}{1 + |z|^2}\right) = 1.$$

Note that

$$\mathbb{E}[Z^\pm] = C \sum_{z=1}^{\infty} 2\pi z \cdot \cos(\pm 2\pi z) \frac{1}{1 + z^2} = 2\pi C \sum_{z=1}^{\infty} \frac{z}{1 + z^2} = \infty \quad (1.13)$$

The divergence of the series arises from the fact that the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges. In other words, $\mathbb{E}[Z]$ does not exist since $\mathbb{E}[Z^+] - \mathbb{E}[Z^-]$ cannot be defined.

However, if in addition of $\xi$ having a symmetric distribution, we assume $\mathbb{E}[|\xi|] < \infty$, then $|Z| \leq |\xi|$ and we conclude that $\mathbb{E}[|Z|] < \infty$, whence $\mathbb{E}[Z] = 0$ by earlier arguments.