Please work out the ten (10) problems stated below; show work and explain reasoning. When not specified, an underlying probability triple \((\Omega, \mathcal{F}, \mathbb{P})\) is always assumed. Throughout \(p\) and \(q\) are positive integers.

Recall

Consider a rv \(X : \Omega \rightarrow \mathbb{R}\) and an \(\mathcal{F}\)-partition \(\{D_i, \ i \in I\}\) where \(I\) is a countable index set. Throughout the events in the partition are assumed to be non-empty. With \(\mathcal{D} = \sigma(D_i, \ i \in I)\), we have defined the conditional expectation of \(X\) given \(\mathcal{D}\) as the rv given by

\[
E[X|\mathcal{D}] \equiv \sum_{i \in I} E[X|D_i] 1[D_i] \tag{1.1}
\]

where \(E[X|D_i]\) is the expectation of \(X\) under the conditional probability distribution of \(X\) given \(D_i\). This definition is well posed if \(X \geq 0\) a.s. or if \(E[|X|] < \infty\).

For easy reference, we introduce the rv \(Z^* : \Omega \rightarrow \mathbb{R} \cup \{-\infty, \infty\}\) given by

\[
Z^* \equiv \sum_{i \in I} E[X|D_i] 1[D_i].
\]

In Exercises 1-4 we assume that the \(\sigma\)-field \(\mathcal{D}\) is generated by an \(\mathcal{F}\)-partition.

1. Explain why the rv \(Z^*\) is well defined as an extended rv if \(X \geq 0\) a.s. or as an \(\mathbb{R}\)-valued rv if \(E[|X|] < \infty\).

2. By direct calculations, show that the rv \(Z\) defined by the right handside of (1.1) satisfies the conditions

\[
E[1[D]Z] = E[1[D]X], \quad D \in \mathcal{D}. \tag{1.2}
\]

Consider the cases \(X \geq 0\) a.s. and \(E[|X|] < \infty\) separately.

In other words, the definition (1.1) given for conditional expectation with respect to the \(\sigma\)-field induced by an \(\mathcal{F}\)-partition coincides with the general definition – More precisely, the rv \(Z^*\) is indeed a representative in the equivalence class \(E[X|\mathcal{D}]\) (as defined at (1.2))!
3. Is the following correct: If the rv $X$ is $\mathcal{D}$-measurable, then

$$
\mathbb{E}[X|\mathcal{D}] = X \quad \text{a.s.}
$$

Use only the definition (1.1)!

4. Consider the $\mathcal{F}$-partitions $\{D_i, \ i \in I\}$ and $\{D_j', j \in J\}$ where $I$ and $J$ are countable index sets. We say that the partition $\{D_j', j \in J\}$ is finer than the partition $\{D_i, i \in I\}$ if for every $i$ in $I$ there exists a subset $J(i)$ such that

$$
D_i = \bigcup_{j \in J(i)} D_j'.
$$

4.a. If the partition $\{D_j', j \in J\}$ is finer than the partition $\{D_i, i \in I\}$, show that $\mathcal{D}$ is a sub-$\sigma$-field of $\mathcal{D}'$ where $\mathcal{D} \equiv \sigma(D_i, i \in I)$ as before and $\mathcal{D}' \equiv \sigma(D_j', j \in J)$.

4.b. By direct calculations, show that

$$
\mathbb{E}[\mathbb{E}[X|\mathcal{D}']|\mathcal{D}] = \mathbb{E}[X|\mathcal{D}] \quad \text{a.s.}
$$

4.c. Show that

$$
\mathbb{E}[\mathbb{E}[X|\mathcal{D}]|\mathcal{D}'] = \mathbb{E}[X|\mathcal{D}] \quad \text{a.s.}
$$

[HINT: Remember Exercise 3.]

Some definitions

Assume a probability triple $(\Omega, \mathcal{F}, \mathbb{P})$. Two collections $\mathcal{A}$ and $\mathcal{B}$ of events (so $\mathcal{A} \subseteq \mathcal{F}$ and $\mathcal{B} \subseteq \mathcal{F}$) are said to be independent (under $\mathbb{P}$) if

$$
\mathbb{P}[A \cap B] = \mathbb{P}[A] \cdot \mathbb{P}[B], \quad A \in \mathcal{A}, \ B \in \mathcal{B}
$$

5. A tricky exercise: Assume the collections $\mathcal{A}$ and $\mathcal{B}$ of events to be fields on $\Omega$. Show that if $\mathcal{A}$ and $\mathcal{B}$ are independent, then the $\sigma$-fields $\sigma(\mathcal{A})$ and $\sigma(\mathcal{B})$ are independent.

6. Let $X : \Omega \to \mathbb{R}^p$ and $Y : \Omega \to \mathbb{R}^q$ be rvs. Show the following fact: The rvs $X$ and $Y$ are independent rvs under $\mathbb{P}$ if and only if the $\sigma$-fields $\sigma(X)$ and $\sigma(Y)$ are independent under $\mathbb{P}$.

Recall

Consider a rv $X : \Omega \to \mathbb{R}$ with $\mathbb{E}[|X|] < \infty$, and let $\mathcal{D}$ denote an arbitrary sub-$\sigma$-field of $\mathcal{F}$. Recall that any representative $Z : \Omega \to \mathbb{R}$ of the conditional expectation of $\mathbb{E}[X|\mathcal{D}]$ is uniquely determined (up to $\mathbb{P}$-a.s. equivalence) by the conditions

$$
\mathbb{E}[1[D]Z] = \mathbb{E}[1[D]X], \quad D \in \mathcal{D}. \quad (1.3)
$$

More precisely:
(i) (Existence) There always exists a $\mathcal{D}$-measurable rv $Z : \Omega \to \mathbb{R}$ with $E[|Z|] < \infty$ such that
\[
E[1[D]Z] = E[1[D]X], \quad D \in \mathcal{D}
\] (1.4)

(ii) (Uniqueness) If the $\mathcal{D}$-measurable rvs $Z_1, Z_2 : \Omega \to \mathbb{R}$ with $E[|Z_1|] < \infty$ and $E[|Z_2|] < \infty$ both satisfy (1.4), namely
\[
E[1[D]Z_k] = E[1[D]X], \quad D \in \mathcal{D}
\] then $Z_1 = Z_2$ $\mathbb{P}$-a.s.

Existence is a consequence of the Radon-Nikodym Theorem. The $\mathcal{D}$-measurable rvs with finite expectation satisfying (1.4) form an equivalence class (under the $\mathbb{P}$-a.s. equivalence); any one of its representatives will be denoted by $E[X|\mathcal{D}]$.

7. Assume now that the rv $X : \Omega \to \mathbb{R}$ is a non-negative rv in the sense that $X \geq 0$ a.s. Can you still define a $\mathcal{D}$-measurable rv which acts as the conditional expectation of $X$ given the $\sigma$-field $\mathcal{D}$ in the sense of (1.4)?

8. Consider a rv $X : \Omega \to \mathbb{R}$ such that $E[|X|] < \infty$. Using only the basic definition (1.4) show the following intuitive statements:

8.a. If $X$ is $\mathcal{D}$-measurable rv, then $E[X|\mathcal{D}] = X$ a.s.

8.b. If $X$ is independent of $\mathcal{D}$ (i.e. the obvious sense that the $\sigma$-fields $\mathcal{D}$ and $\sigma(X)$ are independent), then $E[X|\mathcal{D}] = E[X]$ a.s.

9. Let $\{X_n, n = 1, 2, \ldots\}$ denote a collection of non-negative rvs which are monotonically non-decreasing, i.e.,
\[
X_n \leq X_{n+1}, \quad n = 1, 2, \ldots
\]

With $\mathcal{D}$ denoting a sub-$\sigma$-field of $\mathcal{F}$, show the following version of the Monotone Convergence Theorem for conditional expectations, namely
\[
E \left[ \lim_{n \to \infty} X_n \bigg| \mathcal{D} \right] = \lim_{n \to \infty} E[X_n|\mathcal{D}] \quad \mathbb{P}$-a.s.

[HINT: Remember Exercise 7]

9. Consider rvs $X : \Omega \to \mathbb{R}^p$, $Y : \Omega \to \mathbb{R}^q$, and $Z : \Omega \to \mathbb{R}^r$, and let the Borel mapping $h : \mathbb{R}^p \to \mathbb{R}$ satisfy $E[|h(X)|] < \infty$. We shall assume that the $\mathbb{R}^{p+r}$-valued rv $(X,Z)$ and the $\mathbb{R}^q$-valued rv $Y$ are independent.

9.a. Give arguments as to why you would expect
\[
E[h(X)|Y,Z] = E[h(X)|Z] \quad \mathbb{P}$-a.s. \quad (1.5)

9.b. Prove this when the $\mathbb{R}^{q+r}$-valued rv $(Y,Z)$ is a discrete rv [HINT: There is no loss of generality in assuming that in the condition $\mathbb{P}[(Y,Z) \in S] = 1$, the countable $S \subseteq \mathbb{R}^{q+r}$
can always be taken of the form $S_y \times S_z$ where $S_y$ and $S_z$ are countable subsets of $\mathbb{R}^q$ and $\mathbb{R}^r$, respectively.

9.c. Recall that

$$\sigma(Y, Z) = \sigma \left( [(Y, Z) \in B], \ B \in \mathcal{B} (\mathbb{R}^{q+r}) \right).$$

To establish (1.5) you need to establish appropriate versions of (1.3) with $D = [(Y, Z) \in B]$ as $B$ ranges in $\mathcal{B} (\mathbb{R}^{q+r})$. Show that it is easy to do so when $D$ is a rectangle of the form

$$D = [Y \in A] \cap [Z \in C], \quad A \in \mathcal{B} (\mathbb{R}^q), \quad C \in \mathcal{B} (\mathbb{R}^r)$$

Any idea/suggestion on how to go from that special case to the general case?

10. Consider a Borel mapping $h : \mathbb{R}^p \times \mathbb{R}^q \rightarrow \mathbb{R}$, and rvs $X : \Omega \rightarrow \mathbb{R}^p$ and $Y : \Omega \rightarrow \mathbb{R}^q$ such that $\mathbb{E} [||h(X, Y)||] < \infty$. Define the mapping $\hat{h} : \mathbb{R}^q \rightarrow \mathbb{R}$ given by

$$\hat{h}(y) = \mathbb{E} [h(X, y)], \quad y \in \mathbb{R}^q.$$ 

This definition is always well posed, and produces a Borel mapping $\mathbb{R}^q \rightarrow \mathbb{R}$.

The following result is very useful in many calculations involving conditional expectations: If the rv $X$ is independent of the $\sigma$-field $\mathcal{D}$ and the rv $Y$ is $\mathcal{D}$-measurable, then it holds that

$$\mathbb{E} [h(X, Y)|\mathcal{D}] = \hat{h}(Y) \quad \mathbb{P}\text{-a.s.} \quad (1.6)$$

10.a. In this statement it is natural to understand the independence of the rv $X$ from the $\sigma$-field $\mathcal{D}$ as the statement that for each Borel set $B \in \mathcal{B} (\mathbb{R}^p)$, the event $[X \in B]$ is independent of any event $D$ in $\mathcal{D}$, namely

$$\mathbb{P} [D \cap [X \in B]] = \mathbb{P} [D] \mathbb{P} [[X \in \mathbb{R}^p] D \in \mathcal{D}$$

Show that this condition implies the independence of the rvs $X$ and $Y$.

10.b. To get a sense as to why the result may indeed be true, show its validity in the special case when the Borel mapping $h : \mathbb{R}^p \times \mathbb{R}^q \rightarrow \mathbb{R}$ has the separable form

$$h(x, y) = \sum_{\ell=1}^L g_{\ell}(x) h_{\ell}(y), \quad x \in \mathbb{R}^p, \quad y \in \mathbb{R}^q$$

where for each $\ell = 1, \ldots, L$ the mappings $g_{\ell} : \mathbb{R}^p \rightarrow \mathbb{R}$ and $h_{\ell} : \mathbb{R}^q \rightarrow \mathbb{R}$ are Borel, and $\mathbb{E} [||g_{\ell}(X)||] < \infty$. 