Please work out the ten (10) problems stated below; show work and explain reasoning. When not specified, an underlying probability triple \((\Omega, \mathcal{F}, P)\) is always assumed. Throughout \(p\) and \(q\) are positive integers.

1. First a definition: A collection \(\mathcal{D}\) of subsets of \(\Omega\) is called a sub-\(\sigma\)-field of \(\mathcal{F}\) if \(\mathcal{D}\) is a \(\sigma\)-field on \(\Omega\) such that \(\mathcal{D} \subseteq \mathcal{F}\).
   
   With \(\mathcal{D}\) a sub-\(\sigma\)-field of \(\mathcal{F}\), we say that a rv \(X : \Omega \to \mathbb{R}^p\) is \(\mathcal{D}\)-measurable if
   
   \[ [X \in B] \in \mathcal{D}, \quad B \in \mathcal{B}(\mathbb{R}^p) \]
   
   (and not merely in \(\mathcal{F}\) as required in the definition of \(X\) as a rv.)

   1.a. List all the rvs \(\Omega \to \mathbb{R}^p\) which are \(\mathcal{D}\)-measurable when \(\mathcal{D}\) is the trivial \(\sigma\)-field \(\mathcal{D} = \{\emptyset, \Omega\}\).

   Let \(\mathcal{D}_1\) and \(\mathcal{D}_2\) be two sub-\(\sigma\)-fields of \(\mathcal{F}\) such that \(\mathcal{D}_1 \subseteq \mathcal{D}_2\) (so \(\mathcal{D}_1\) is a sub-\(\sigma\)-field of \(\mathcal{D}_2\)).

   1.b. Show that a rv \(X : \Omega \to \mathbb{R}^p\) which is \(\mathcal{D}_1\)-measurable is also \(\mathcal{D}_2\)-measurable.

   1.c. Consider now a rv \(X : \Omega \to \mathbb{R}^p\) which is \(\mathcal{D}_1\)-measurable. Is it automatically \(\mathcal{D}_2\)-measurable? Either prove or give a counterexample.

2. Let the sub-\(\sigma\)-field \(\mathcal{D}\) of \(\mathcal{F}\) be generated by some rv \(Y : \Omega \to \mathbb{R}^q\) in the (usual) sense that

   \[ \mathcal{D} = \sigma ([Y \in C], \quad C \in \mathcal{B}(\mathbb{R}^q)) \]

   We shall then write \(\mathcal{D} = \sigma(Y)\).

   2.a. Show that if the rv \(X : \Omega \to \mathbb{R}\) is of the form

   \[ X = g(Y) \]

   for some Borel mapping \(g : \mathbb{R}^q \to \mathbb{R}\), then the rv \(X\) is necessarily \(\sigma(Y)\)-measurable.

   2.b. The converse is true: Think about how you would prove this fact – A proof will be provided in the Lecture Notes.
3. A rv $X : \Omega \to \mathbb{R}$ is said to have a symmetric probability distribution if the rvs $X$ and $-X$ have the same probability distribution (under $\mathbb{P}$).

3.a. Give necessary and sufficient conditions on $F_X : \mathbb{R} \to [0,1]$ for the rv $X$ to have a symmetric probability distribution.

3.b. If $X$ has a symmetric probability distribution, show that $\mathbb{E}[X] = 0$ whenever $\mathbb{E}[X]$ is well defined and finite.

3.c. If $X$ has a symmetric probability distribution, can $\mathbb{E}[X^+]$ and $\mathbb{E}[X^-]$ be different?

3.d. Fix $a > 0$. With the rv $X$, we associate the rv $Y_a : \Omega \to \mathbb{R}$ given by

$$Y_a = \begin{cases} X & \text{if } |X| \leq a \\ -X & \text{if } a < |X|. \end{cases}$$

If $X$ has a symmetric probability distribution, show that the rv $Y_a$ has the same distribution as the rv $X$. This problem is often formulated with $X \sim \mathcal{N}(0, 1)$ but the result holds more generally and requires very little computation. Again the power of probabilistic thinking at work!

4. Let $X, Y : \Omega \to \mathbb{R}$ be a pair of independent rvs. Assume that the rvs are exponentially distributed in the sense that

$$\mathbb{P}[X \leq t] = \mathbb{P}[Y \leq t] = 1 - e^{-\lambda t^+}, \quad t \in \mathbb{R}$$

for some $\lambda > 0$ with $t^+ = \max(t, 0)$ for each $t$ in $\mathbb{R}$. Using direct integration arguments\(^\dagger\)

4.a. Compute $\mathbb{P}[X \leq Y]$.


4.c. Compute

$$\mathbb{P}[Z \leq z], \quad z \in \mathbb{R}$$

where we have set

$$Z = \begin{cases} \frac{X}{X+Y} & \text{if } X + Y > 0 \\ 0 & \text{if } X + Y \leq 0. \end{cases}$$

We shall revisit these calculations after discussing conditioning with respect to rvs which are absolutely continuous.

5. Consider an $\mathcal{F}$-partition $\{D_i, i \in I\}$ where $I$ is a countable index set: For each $i$ in $I$, the set $D_i$ is an event in $\mathcal{D}$ with

$$D_i \cap D_j = \emptyset, \quad i \neq j, \quad i, j \in I$$

\(^\dagger\)HINT: Recall that the $\mathbb{R}^2$-valued rv $(X, Y)$ is of continuous type [WHY?] and its probability density function is given by . . .
and

\[ \bigcup_{i \in I} D_i = \Omega. \]

For simplicity we can assume that the events in the partition are non-empty.

Set \( D = \sigma(D_i, i \in I) \) and recall from Quiz # 5 that any element \( D \) of \( D \) is necessarily of the form

\[ D = \bigcup_{j \in J} D_j \]

for some countable subset \( J \subseteq I \) (possibly empty if \( D = \emptyset \) or \( J = I \) if \( D = \Omega \)).

5.a. Show that any \( D \)-measurable rv \( X : \Omega \to \mathbb{R} \) is necessarily constant on the “atoms” \( \{D_i, i \in I\} \) that generate \( D \).

5.b. If \( \{a_i, i \in I\} \) is collection of vectors in \( \mathbb{R}^p \), define the mapping \( X : \Omega \to \mathbb{R}^p \) by

\[ X(\omega) = a_i \quad \text{if} \quad \omega \in D_i \quad i \in I \]

i.e.,

\[ X = \sum_{i \in I} a_i 1[D_i] \]

where \( 1[D_i] \) is the indicator of the event \( D_i \). Is \( X \) a rv? If so, show that it is of discrete type and determined its pmf.

6. Let \( X, Y : \Omega \to \mathbb{R} \) be rvs with finite second moments, namely \( \mathbb{E}[X^2] < \infty \) and \( \mathbb{E}[Y^2] < \infty \). The covariance \( \text{Cov}[X, Y] \) between the rvs \( X \) and \( Y \) is defined by

\[ \text{Cov}[X, Y] = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]. \]

The rvs \( X \) and \( Y \) are said to be uncorrelated if \( \text{Cov}[X, Y] = 0 \).

6.a. Explain why this quantity is well defined under the assumption that the rvs \( X \) and \( Y \) have each a finite second moment [HINT: Remember the Cauchy-Schwarz inequality].

6.b. Under the assumption that the rvs \( X \) and \( Y \) have each a finite second moment, show that the rvs \( X \) and \( Y \) are uncorrelated when the rvs \( X \) and \( Y \) are independent.

6.c. Through a counterexample show that even under the assumption that the rvs \( X \) and \( Y \) have each a finite second moment, the rvs \( X \) and \( Y \) can be uncorrelated without being independent [HINT: The classical counterexample: \( X = \cos(2\pi\Theta) \) and \( Y = \sin(2\pi\Theta) \) where \( \Theta \) is uniformly distributed on the interval \((0, 1)\)].

8. In this problem we briefly discuss how \( \mathcal{F} \)-partitions are induced by discrete rvs. Consider a discrete rv \( Y : \Omega \to \mathbb{R}^q \). By definition there exists a countable subset \( S \subseteq \mathbb{R}^q \) such that \( \mathbb{P}[Y \in S] = 1 \). For ease of notation, with \( I \) countable we represent \( S \) as \( S = \{y_i, i \in I\} \) where the elements are distinct. So far, by the definition of discrete rvs, we can only assert that the event

\[ \Omega_Y \equiv \bigcup_{i \in I} [Y = y_i] \]

has probability one, or equivalently, that the complement \( \Omega_Y^c \) has zero probability. Nothing precludes the set of values

\[ \{Y(\omega), \omega \notin \Omega_Y\} \]
to form an uncountable set. Only when that set is empty, will the collection \( \{ [Y = y_i], \ i \in I \} \) be an \( \mathcal{F} \)-partition.

8.a. Give an example of a discrete rv \( Y : \Omega \to \mathbb{R}^q \) for which \( \Omega_Y \neq \Omega \) and the set \( \{ Y(\omega), \ \omega \notin \Omega_Y \} \) is uncountable in \( \mathbb{R}^q \)!

8.b. With \( b \) an element not in \( S \), now define the rv \( Y_b : \Omega \to \mathbb{R}^q \) given by

\[
Y_b(\omega) \equiv \begin{cases} 
Y(\omega) & \text{if } \omega \in \Omega_Y \\
b & \text{if } \omega \notin \Omega_Y
\end{cases}
\]

Show that the collection \( \{ [Y = b], \ [Y = y_i], \ i \in I \} \) is now an \( \mathcal{F} \)-partition.

8.c. Show that the rvs \( Y \) and \( Y_b \) have the same probability distribution under \( \mathbb{P} \).

8.d. If \( X : \Omega \to \mathbb{R}^p \) is another rv, show that the pairs \((X,Y)\) and \((X,Y_b)\) have the same probability distribution under \( \mathbb{P} \).

9.

Recall that if \( X : \Omega \to \mathbb{R} \) is a discrete rv, then the expectation \( \mathbb{E} [X] \) can be defined as

\[
\mathbb{E} [X] = \sum_{x \in S} x \mathbb{P} [X = x]
\]

where \( S \) is the support of \( X \), namely that countable subset of \( \mathbb{R} \) with the property that \( \mathbb{P} [X \in S] = 1 \). Implicit in this definition is that either (i) \( \sum_{x \in S} |x| \mathbb{P} [X = x] < \infty \) or (ii) \( \sum_{x \in S} x^+ \mathbb{P} [X = x] = \infty \) with \( \sum_{x \in S} x^- \mathbb{P} [X = x] < \infty \) or (iii) \( \sum_{x \in S} x^- \mathbb{P} [X = x] = \infty \) with \( \sum_{x \in S} x^+ \mathbb{P} [X = x] < \infty \). Condition (i) automatically applies when \( S \) is a finite set.

Compute the expectation

\[
\mathbb{E} \left[ \frac{1}{1 + Y^+} \right]
\]

when the rv \( Y : \Omega \to \mathbb{R} \) is

9.a. a binomial rv Bin\((n;p)\) with \( n = 1, 2, \ldots \) and \( 0 < p < 1 \), i.e.,

\[
\mathbb{P} [Y = k] = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n.
\]

9.b. a Poisson rv Pois\((\lambda)\) with \( \lambda > 0 \), i.e.,

\[
\mathbb{P} [Y = k] = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, \ldots
\]

9.c. a geometric rv Geo\((p)\) with \( 0 < p < 1 \), i.e.,

\[
\mathbb{P} [Y = k] = p (1-p)^k, \quad k = 0, 1, \ldots
\]

In each case explain why the expectation \( \mathbb{E} \left[ \frac{1}{1 + Y} \right] \) always exists.

10.

An \( \mathbb{N} \)-valued rv \( Z \) is said to be a Poisson rv with parameter \( \lambda > 0 \) if its pmf is given by

\[
\mathbb{P} [Z = k] = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, \ldots, \lambda > 0
\]
We shall write $Z =_{st} \text{Poi}(\lambda)$.

**10.a.** Let $X$ and $Y$ be two independent Poisson rvs, namely $X =_{st} \text{Poi}(\lambda)$ and $Y =_{st} \text{Poi}(\mu)$ with $\lambda, \mu > 0$. Show that the rv $Z = X + Y$ is also a Poisson rv. Identify its parameter. Generalize to $K$ mutually independent Poisson rvs $X_1, \ldots, X_K$ with

$$X_k =_{st} \text{Poi}(\lambda_k) \quad \lambda_k > 0 \quad k = 1, \ldots, K.$$  

Carefully explain your reasoning.

**10.b.** Let $N$ be a Poisson rv, and let $\{B_n, \ n = 1, 2, \ldots\}$ be a collection of Bernoulli rvs with

$$P[B_n = 1] = 1 - P[B_n = 0] = p, \quad n = 1, 2, \ldots$$

$0 < p < 1$. If the rvs $\{N, B_n, \ n = 1, 2, \ldots\}$ are mutually independent, show that the rvs $X$ and $Y$ defined by

$$X := \sum_{i=1}^{N} B_i \quad \text{and} \quad Y := \sum_{i=1}^{N} (1 - B_i)$$

are independent with $X$ and $Y$ Poisson rvs with parameters $\lambda p$ and $\lambda(1 - p)$, respectively.

**10.c.** Can you use the result of Part b to provide an alternative solution to Part a? Explain! Again a case of probabilistic reasoning at work!