Please work out the ten (10) problems stated below; show work and explain reasoning. When not specified, an underlying probability triple \((\Omega, \mathcal{F}, P)\) is always assumed.

1. With positive integer \(p\), recall that the Borel \(\sigma\)-field on \(\mathbb{R}^p\) is the smallest \(\sigma\)-field on \(\mathbb{R}^p\) generated by the collection \(\mathcal{O}(\mathbb{R}^p)\) of all open sets in \(\mathbb{R}^p\), i.e.,

\[
\mathcal{B}^{\mathbb{R}^p} = \sigma(\mathcal{O}(\mathbb{R}^p)).
\]

As explained in class, it is often possible to find other collections \(\mathcal{G}\) of subsets of \(\mathbb{R}^p\) which also generate the Borel \(\sigma\)-field \(\mathcal{B}(\mathbb{R}^p)\), i.e.,

\[
\mathcal{B}(\mathbb{R}^p) = \sigma(\mathcal{G}). \quad (1.1)
\]

1.a. Show that (1.1) holds with \(\mathcal{G} = \mathcal{R}_{\text{open}}(\mathbb{R}^p)\) where \(\mathcal{R}_{\text{open}}(\mathbb{R}^p)\) is the collection of all finite open rectangles, i.e.,

\[
\mathcal{R}_{\text{open}}(\mathbb{R}^p) \equiv \left\{ I_1 \times \ldots \times I_p, \quad I_k \in \mathcal{I}(\mathbb{R}) \right\}_{k = 1, \ldots, p}
\]

where

\[
\mathcal{I}(\mathbb{R}) = \left\{ (a, b) : \begin{array}{c}
a < b \\
a, b \in \mathbb{R}
\end{array} \right\}
\]

HINT: You will need to use the following fact: For any open set \(U\) in \(\mathbb{R}^p\) there exists a countable family of open rectangles \(\{R_i, \ i \in I\}\) in \(\mathcal{R}_{\text{open}}(\mathbb{R}^p)\) with countable \(I\) such that \(U = \bigcup_{i \in I} R_i\). [Try to prove it! It is the analog of a similar fact encountered in one dimension].

1.b. Show that (1.1) holds with \(\mathcal{G} = \mathcal{R}_{\text{SW}}(\mathbb{R}^p)\) where \(\mathcal{R}_{\text{SW}}(\mathbb{R}^p)\) is the collection of all closed Southwest rectangles, i.e.,

\[
\mathcal{R}_{\text{SW}}(\mathbb{R}^p) \equiv \left\{ I_1 \times \ldots \times I_p, \quad I_k = (-\infty, a_k] \right\}_{k = 1, \ldots, p}
\]

\[a_k \in \mathbb{R}\]
If you have difficulties establishing Part a, just assume it to be true and proceed!

2. Let $S$ be an arbitrary set equipped with a $\sigma$-field $\mathcal{S}$. Recall that a mapping $g : S \rightarrow \mathbb{R}^p$ is said to be a Borel mapping if the conditions

$$g^{-1}(B) \equiv \{ s \in S : g(s) \in B \} \in \mathcal{S}, \quad B \in \mathcal{B}(\mathbb{R}^p)$$

are all satisfied.

Let $\mathcal{G}$ denote a collection of subsets of $\mathbb{R}^p$ which generates the Borel $\sigma$-field $\mathcal{B}(\mathbb{R}^p)$, i.e.,

$$\mathcal{B}(\mathbb{R}^p) = \sigma (\mathcal{G}).$$

Show that the mapping $g : S \rightarrow \mathbb{R}^p$ is a Borel mapping if and only if the weaker set of conditions

$$g^{-1}(G) \equiv \{ s \in S : g(s) \in G \} \in \mathcal{S}, \quad G \in \mathcal{G}$$

holds.

One implication is trivial, namely that (??) implies (??). To prove the reverse implication here is a hint: Consider the collection $\mathcal{E}_g$ of subsets of $\mathbb{R}^p$ given by

$$\mathcal{E}_g \equiv \{ E \subseteq \mathbb{R}^p : g^{-1}(E) \in \mathcal{S} \}.$$ 

Show that $\mathcal{E}_g$ is a $\sigma$-field on $\mathbb{R}^p$. Under condition (??) the inclusion $\mathcal{G} \subseteq \mathcal{E}_g$ holds, hence ...

3. With positive integer $p$, recall that an $\mathbb{R}^p$-valued rv is any mapping $\Omega \rightarrow \mathbb{R}^p$ such that

$$X^{-1}(B) = [X \in B] \in \mathcal{F}, \quad B \in \mathcal{B}(\mathbb{R}^p).$$

where $\mathcal{B}(\mathbb{R}^p) = \sigma (\mathcal{O}(\mathbb{R}^p))$ (with $\mathcal{O}(\mathbb{R}^p)$ denoting the collection of all open sets in $\mathbb{R}^p$).

Using the results in Problems 1 and 2 show that a mapping $\Omega \rightarrow \mathbb{R}^p$ is an $\mathbb{R}^p$-valued rv if and only if it satisfies the conditions

$$X^{-1}(R) = [X \in R] \in \mathcal{F}, \quad R \in \mathcal{R}_{SW}(\mathbb{R}^p).$$

4. Use the information gleaned in Problem 3 to conclude to the following very important facts: Consider a collection of rvs $\{X_i, \ i \in I\}$ all defined on some probability triple $(\Omega, \mathcal{F}, P)$ where for each $i$ in $I$ the rv $X_i$ is $\mathbb{R}^{p_i}$-valued rv for some positive integer $p_i$.

Recall that with $I$ finite, the rvs $\{X_i, \ i \in I\}$ are mutually independent if for each selection of $(B_i \in \mathcal{B}(\mathbb{R}^{p_i}), \ i \in I)$ the events

$$\{ [X_i \in B_i], \ i \in I \}$$

are mutually independent.
4.a. With $I$ finite, show that this definition is equivalent to
\[ P[\bigcap_{i \in I} [X_i \in B_i]] = \prod_{i \in I} P[X_i \in B_i], \quad B_i \in \mathcal{B}(\mathbb{R}^p) \]

4.b. With $I$ finite, show that this definition is also equivalent to
\[ F_{(X_i, i \in I)}(x_i, i \in I) = \prod_{i \in I} F_{X_i}(x_i), \quad x_i \in \mathbb{R} \]

4.c. When $I$ is an arbitrary index set (possibly uncountable), how would you define the mutual independence of the rvs $\{X_i, i \in I\}$? Carefully explain your answer!

5. With positive integer $p$, consider an $\mathbb{R}^p$-valued rv $X$, i.e., a mapping $X : \Omega \to \mathbb{R}^p$ which satisfies the conditions (??).

5.a. Define $P_X : \mathcal{B}(\mathbb{R}^p) \to [0,1]$ by
\[ P_X [B] \equiv P[X \in B], \quad B \in \mathcal{B}(\mathbb{R}^p). \]
Is this definition well posed? Explain. Show that $P_X : \mathcal{B}(\mathbb{R}^p) \to [0,1]$ is a probability measure on $(\mathbb{R}^p, \mathcal{B}(\mathbb{R}^p))$.

5.b. Explain in what sense the triple $(\mathbb{R}^p, \mathcal{B}(\mathbb{R}^p), P_X)$ can be interpreted as a probability model for an experiment, say $\mathcal{E}_X$, derived from $\mathcal{E}$ (the random experiment that was modeled by the triple $(\Omega, \mathcal{F}, P)$) and induced by $X$. Is there a “problem” if it happens that $P_X [B] = 0$ for some Borel subset $B$ in $\mathbb{R}^p$? Explain!

6. Consider the situation of Section 2.3 of the Lecture Notes: The experiment $\mathcal{E}$ consists in repeating a coin toss under "identical and independent conditions" with a fair coin (so that the likelihood of occurrence of Head is the same as that of Tail). It is convenient to take the sample space $\Omega$ to be $\{0,1\}^{\mathbb{N}_0}$, i.e.,
\[ \Omega = \{\omega = (\omega_1, \omega_2, \ldots) : \omega_k \in \{0,1\}, \ k = 1,2,\ldots\} \]
with the understanding that $\omega_k = 1$ (resp. $\omega_k = 0$) if the $k^{th}$ toss yields Head (resp. Tail).

On the way to constructing $\mathcal{F}$ and $P$, it was argued as follows: It is natural to require that for any $n = 1,2,\ldots$, any collection of outcomes determined by the first $n$ tosses should be an event in $\mathcal{F}$ – After all one should expect that the model we are seeking to construct would also contain a model for each of the finite toss experiments. In particular, with any given binary sequence $(b_1,\ldots,b_n)$ of length $n$, consider
\[ F_n(b_1,\ldots,b_n) \equiv \left\{\omega = (\omega_1,\omega_2,\ldots) \in \Omega : \omega_k = b_k \right\} \quad (1.6) \]
It is plain that $\mathcal{F}$ must at least contain these events which are determined by a finite number of coin tosses, namely $F_n(b_1,\ldots,b_n)$ should be an element in $\mathcal{F}$.
Fairness (which is essentially a uniformity condition) requires that
\[ P[F_n(b_1, \ldots, b_n)] = 2^{-n} \tag{1.7} \]
since \( P[F_n(b_1, \ldots, b_n)] \) should not depend on \((b_1, \ldots, b_n)\) and there are \(2^n\) distinct and non-overlapping sets of the form \((??)\) with
\[ \bigcup_{(b_1, \ldots, b_n) \in \{0,1\}^n} F_n(b_1, \ldots, b_n) = \Omega. \]

It was argued that
\[ \mathcal{F} \equiv \sigma \left\{ F_n(b_1, \ldots, b_n), \quad b_1, \ldots, b_n \in \{0,1\}, \quad n = 1, 2, \ldots \right\}. \]

Now consider the sets \( \{A_n, n = 1, 2\ldots\} \) defined by
\[ A_n \equiv \{ \omega = (\omega_1, \omega_2, \ldots) \in \Omega : \omega_n = 1 \}, \quad n = 1, 2, \ldots \]

6.a. Show that for each \( n = 1, 2, \ldots \), \( A_n \) is an event in \( \mathcal{F} \) - It is essentially the event where the \( n^{th} \) toss is a Head.

6.b. Show that
\[ P[A_n] = \frac{1}{2}, \quad n = 1, 2, \ldots \]

6.c. Show that the events \( \{A_n, n = 1, 2\ldots\} \) are mutually independent, thereby recovering the qualifier that the tosses are “independent.” Note that this is a condition (??) alone! In other words, here successive tosses under identical conditions implies independence! Note that this argument will not work when the coin is not a fair coin; see Problem 7.

7. Consider the following random experiment: A coin is tossed infinitely many times under “identical and independent” conditions; the outcome of each toss is recorded as either Head or Tail. It is assumed that on a single toss the likelihood of observing Head is \( p \) (with \( 0 < p < 1 \)).

7.a. Develop a probability model \((\Omega, \mathcal{F}, P)\) for this experiment.
7.b. What is the probability that a Head eventually occurs?
7.c. What is the probability that a Tail eventually occurs?
7.d. What is the probability that two Heads are eventually observed?

8. Imagine that \( N \) tickets are sold in a lottery, of which \( W \) are winning tickets. Mr. Noone buys \( K \) tickets. Assume that \( W + K < N \).

8.a. Construct a probability model \((\Omega, \mathcal{F}, P)\) that would model this situation under the natural assumption that the tickets are indeed sold at random.
8.b. Use this probability model to compute the probability that Mr. Noone has bought at least one winning ticket.
Your cupboard contains six cups and six saucers. There are two blue cups and two blue saucers, two red cups and two red saucers, two white cups and two white saucers. As you are setting the table for a dinner party, you randomly assign cups to saucers.

9.a. Construct a probability model \((\Omega, \mathcal{F}, \mathbb{P})\) that would model this situation.

9.b. Compute the probability that none of the six cups is assigned to a saucer of the same color!