1. Let \( \Omega = \{0, 1, 2\} \).
Here \( \Omega \) is a discrete sample space.
Construct \( \mathcal{F} \) as
\[
\mathcal{F} = \{\emptyset, \{0\}, \{1, 2\}, \{0, 1, 2\}\}
\]
It is straightforward to check that \( \mathcal{F} \) is a \( \sigma \)-field.
It is also clear that \( \mathcal{F} \) is neither \( 2^\Omega \) nor \( \{\emptyset, \{0, 1, 2\}\} \) (i.e., trivial \( \sigma \)-field).

Now if we try to assign probability measure \( P : \mathcal{F} \to [0, 1] \), we have to assign probability values to each of the elements of \( \mathcal{F} \).

Now it is clear that with such a measure, we can’t say anything about \( P[\{1\}] \) or \( P[\{2\}] \), i.e., we cannot assign probability values to the singletons, which is important for a probability model.

Hence this \( \sigma \)-field is poorly equipped for a probability model.
Mutual independence: \( P \left( \bigcap_{j \in J} A_j \right) = \prod_{j \in J} P \left( A_j \right) \) \( j \leq J \leq \{1, \ldots, n\} \)

Pairwise independence: \( P \left( A_i \cap A_j \right) = P \left( A_i \right) P \left( A_j \right) \) \( i, j = 1, \ldots, n \)

The \( n(n-1) \) conditions of (1.2) are a subset \( \frac{n}{2} \) of \( 2^n - (n+1) \) conditions of (1.1).

\( \therefore \) Mutual independence \( \Rightarrow \) Pairwise independence

The converse does NOT hold.

Counter example: unbiased

Let a \( \{0,1\} \) faced die is tossed twice.

\( F = \{0,1\}^2 \)

\( P \) is such defined that \( P \) of each singleton is \( \frac{1}{4} \).

Define three events \( B_1, B_2, B_3 \) as follows:

\( B_i \): \( i \)-th outcome is '1' for \( i = 1, 2 \)

\( B_3 \): odd no. of 1's occurred in the two tosses

Then clearly \( P \left( B_i \right) = \frac{1}{2} \) and

\( P \left( B_1 \cap B_2 \right) = P \left( B_1 \cap B_3 \right) = P \left( B_2 \cap B_3 \right) = \frac{1}{4} \)

So \( \{B_1, B_2, B_3\} \) are pairwise independent.
But \[ P(B_1 \cap B_2 \cap B_3) = 0 \neq P(B_1)P(B_2)P(B_3) \]
\( \therefore \) \{\( B_1, B_2, B_3 \)\} are not mutually independent.
The only condition that holds:

\[ P[\bigwedge_{i \in I} F_i A_i] = \prod_{i \in I} P[A_i] \]  \quad \text{(1.3)}

But the other conditions of (1.1) doesn't.
Then \( \bigwedge_{i \in I} A_i \) is NOT independent of \( A \).

Counterexample:
Let \( \Omega = \{0,1\}^3 \) with an uniform probability measure on it. (i.e., each of the singleton has probability \( \frac{1}{8} \))
\( n = 2^3 \) since \( \Omega \) is discrete.
This random experiment \( E : (\Omega, 2^n, P) \)
can be thought as tossing of a \( \{0,1\} \) faced unbiased coin 3 times in a row.
Now define events \( B_1, B_2 \) & \( B_3 \) as follows:

\( B_1 \) is the event that among the three tosses, we get more '0' than '1' (i.e., we get \( (0,0,0), (0,0,1), (0,1,0) \) or \( (1,0,0) \)).
Clearly \( P[B_1] = \frac{1}{2} \)

\( B_2 \) is the event that we get '1' in the last of the three tosses (i.e., the union of \( (0,0,1), (0,1,1), (1,0,1) \) & \( (1,1,1) \)).
Clearly \( P[B_2] = \frac{1}{2} \)

\( B_3 \) is the event that the second toss return '0' (i.e., \( \{(0,0,0), (0,0,1), (1,0,0), (1,1,0)\} \))
Clearly \( P[B_3] = \frac{1}{2} \)
Clearly, \[ P[B_1 \cap B_2 \cap B_3] = P[\emptyset] = \frac{1}{8} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = P[B_1] \cdot P[B_2] \cdot P[B_3] \]

\[ \implies (1, 3) \text{ is satisfied.} \]

Now, \[ B_1 \cap B_2 = \{ (0, 0, 1) \} \]

\[ \implies P[B_1 \cap B_2] = \frac{1}{8} \]

\[ \therefore P[(\emptyset \cap B_1) \cap B_3] = P[B_1 \cap B_2 \cap B_3] = \frac{1}{8} \neq \frac{1}{8} \times \frac{1}{2} = P[B_1 \cap B_2] \cdot P[B_3] \]

\[ \therefore B_1 \cap B_2 \text{ is not independent w.r.t. } B_3. \]
Suppose \( \{ A_n, n = 1, 2, \ldots \} \) is countably infinite collection of events. Now if we only consider cases where \( 0 < |J| < \infty \) in Condition (1.4), it can be written as
\[
P \left[ \bigcap_{k=1}^{K} A_{n_k} \right] = \prod_{k=1}^{K} P[A_{n_k}]
\]
where \( \{ A_{n_k} : k = 1, \ldots, K \} \subseteq \{ A_n, n = 1, 2, 3, \ldots \} \)
and each finite subsequence
\[
\bigwedge_{k=1}^{K} A_{n_k} : \{1, \ldots, K\} \rightarrow \mathbb{N}
\]
Now from continuity lemma \( K \)
\[
P \left[ \bigcap_{k=1}^{\infty} A_{n_k} \right] = \lim_{K \to \infty} \prod_{k=1}^{K} P[A_{n_k}]
\]
Since \( B_K = \bigcap_{k=1}^{K} A_{n_k} \) is a decreasing sequence of sets,
i.e., \( B_K \supseteq B_{K+1} \) and \( \bigcap_{k=1}^{K} A_{n_k} = \bigcap_{k=1}^{K} B_{n_k} \)
So it is unnecessary to include \( |J| \)
countably infinite case as it doesn't add any extra condition.

If \( I \) is arbitrary (i.e., might be uncountable), if we define mutual indep. as (1.5) where \( |J| \) is possibly uncountable then it is possible that \( \bigcap_{j \in J} A_j \in \mathcal{F} \)
since the definition of \( \mathcal{A} \)-field says countable union only.
So, the defn (1.5) does NOT make sense. So we must restrict $|J| < H_0$ which in turn from part 1 will be equivalent to restrict $|J| < \infty$. 

\[ \therefore \{ A_i, i \in I \} \text{ are mutually independent} \]

(irrespective of whether $|I| \leq H_0$ or $|I| > H_0$)

\[
P[ \bigcap_{j \in J} A_j ] = \prod_{j \in J} P[A_j] \quad \forall J \subseteq I \quad 0 < |J| < \infty
\]

\[5. \] See Sec 0.1 Solution in Canvas.
We know from Boole's that

\[ P \left[ \bigcup_{i=1}^{m} A_i \right] \leq \sum_{i=1}^{m} P[A_i] \]

We need to show

\[ P \left[ \bigcup_{i \in I} A_i \right] \leq \sum_{i \in I} P[A_i] \]

when \( |I| < |I_0| \),

which means

\[ P \left[ \bigcup_{i=1}^{\infty} A_i \right] \leq \sum_{i=1}^{\infty} P[A_i] \]

for some enumeration of \( I \).

Now let's define

\[ B_n \equiv \bigcup_{i=1}^{n} A_i \quad \text{with} \quad B_1 \equiv A_1 \]

Therefore

\[ B_{n+1} \supseteq B_n \]

and

\[ \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i \]

\[ \therefore P \left[ \bigcup_{i=1}^{\infty} A_i \right] = P \left[ \bigcup_{i=1}^{\infty} B_i \right] \]

\[ = \lim_{i \to \infty} P[B_i] \]

\[ = \lim_{n \to \infty} P[B_n] \]

\[ = \lim_{n \to \infty} P \left[ \bigcup_{i=1}^{n} A_i \right] \]
\[
\leq \lim_{n \to \infty} \sum_{i=1}^{n} P[A_i] \quad \text{[From finite inequality]}
\]

\[
= \sum_{i=1}^{\infty} P[A_i]
\]
4. Bonferroni's inequality:

\[ P \left( \bigcup_{i=1}^{m} A_i \right) \geq \sum_{i=1}^{m} P[A_i] - \sum_{1 \leq i < j \leq m} P[A_i \cap A_j] \]

**Proof:**

For \( n = 2 \), this is trivially true (with an equality)

Let the inequality be true for \( n = m \geq 2 \)

\[ P \left[ \bigcup_{i=1}^{m+1} A_i \right] = P \left[ \bigcup_{i=1}^{m} A_i \right] + P[A_{m+1}] \]

\[ - P \left[ \bigcup_{i=1}^{m} (A_i \cap A_{m+1}) \right] \]

\[ = P \left[ \bigcup_{i=1}^{m} A_i \right] + P[A_{m+1}] - P \left[ \bigcup_{i=1}^{m} (A_i \cap A_{m+1}) \right] \]

\[ \geq P \left[ \bigcup_{i=1}^{m} A_i \right] + P[A_{m+1}] - \sum_{i=1}^{m} P[A_i \cap A_{m+1}] \]

\[ \geq \sum_{i=1}^{m} P[A_i] - \sum_{1 \leq i < j \leq m} P[A_i \cap A_j] + P[A_{m+1}] \]

\[ - \sum_{i=1}^{m} P[A_i \cap A_{m+1}] \]

\[ \geq \sum_{i=1}^{m} P[A_i] - \sum_{1 \leq i < j \leq m} P[A_i \cap A_j] \]

\[ \geq \sum_{i=1}^{m+1} P[A_i] - \sum_{1 \leq i < j \leq m} P[A_i \cap A_j] \]

\[ \therefore \text{The result is true by induction!} \]
A, B, E, F mutually independent.

1. \( P[\neg A \cap B] = P[A] P[B] \) \[\text{trivially satisfied}\]

2. \( P[E^c \cap F^c] = P[(E \cup F)^c] \)
   \[= 1 - P[E \cup F] \]
   \[= (1 - P[E])(1 - P[F]) \]
   \[= P[E^c] P[F^c] \]

3. \( P[A \cap E^c] = P[A \setminus E] \)
   \[= P[A] - P[A \cap E] \]
   \[= P[A] - P[A] P[F] \]
   \[= P[A] (1 - P[F]) \]
   \[= P[A] P[E^c] \]

4. \( P[B \cap E^c] = P[B] P[E^c] \)

5. \( P[A \cap F^c] = P[A] P[F^c] \)

6. \( P[B \cap F^c] = P[B] P[F^c] \) \[\text{similar to 3.}\]

7. \( P[A \cap B \cap E^c] = P[A] P[B] P[E^c] \)

8. \( P[A \cap B \cap F^c] = P[A] P[B] P[F^c] \) \[\text{take } (A \cap B) \text{ and proceed like 3.}\]

9. \( P[A \cap E^c \setminus F^c] \)
   \[= P[A] - P[A \cap (E \cup F)] \]
   \[= P[A] - P[(A \setminus E) \cup (A \setminus F)] \]
   \[= P[A] - P[A \setminus E] - P[A \setminus F] + P[(A \setminus E) \cap (A \setminus F)] \]
   \[= P[A] - P[A \setminus E] - P[A \setminus F] + P[A \setminus E \cap A \setminus F] \]
   \[= P[A] (1 - P[E] - P[F] + P[E] P[F]) \]
   \[= P[A] P[E^c] P[F^c] \]
10. \[ P[A \cap E^c \cap F^c] = P[B]P[E]P[F^c] \]
   [Similar to 9.]

11. \[ P[A \cap B \cap E^c \cap F^c] = P[A \cap B] - P[(A \cap B) \cap (E \cup F)] \]
   [Take \([A \cap B]\) as single event and proceed like 9.]

\[ P[A, B, E^c \cap F^c] \]

Since all \(2^4 - 4 - 1 = 11\) conditions are satisfied:

ii) \[ P[(A \cup B) \cap E^c] = P[A \cap E^c] + P[B \cap E^c] - P[A \cap B \cap E^c] \]
\[= (P[A] + P[B] - P[A \cap B])P[E^c] \]
\[= P[A \cup B]P[E^c]\]

Similarly 2. \[ P[(A \cup B) \cap F^c] = P[A \cap F^c] + P[B \cap F^c] - P[A \cap B \cap F^c] \]

3. \[ P[E^c \cap F^c] = P[E^c]P[F^c] \]
   [From part (i), 2.]

4. \[ P[(A \cup B) \cap E^c \cap F^c] \]
   \[= P[A \cup B] - P[(A \cup B) \cap (E \cup F)] \]
   Now take \([A \cup B]\) as a single event and proceed like part (i), 9.

So all \(2^3 - (3 + 1) = 4\) conditions are satisfied.

\[ P[(A \cup B) \cap (E \cup F)] \]

\[= P[(A \cup B) \cap E] + P[(A \cup B) \cap F] - P[(A \cup B) \cap (E \cap F)] \]
\[= P[A \cup B]P[E] + P[A \cup B]P[F] \]
   [By part (iii)]
\[ E_n(b_1, \ldots, b_n) = \{ \omega \in \Omega : \omega_i = b_i, i = 1, \ldots, n \} \]
\[ \mathbb{P}[E_n(b_1, \ldots, b_n)] = 2^{-n} \]

Let \( A_n = \{ \omega \in \Omega : \omega_n = 1 \} \), \( n = 1, 2, \ldots \)

(i) \[ \mathbb{P}[A_n] = \mathbb{P}[\{ \omega \in \Omega : \omega_n = 1 \}] = \sum \mathbb{P}[E_n(b_1, \ldots, b_{n-1}, 1)] \]
\[ \{b_1, \ldots, b_{n-1}\} \in \{0,1\}^{n-1} \]
\[ = 2^{(n-1)} 2^{-n} = \frac{1}{2} \]

We have to sum over all those \( E_n \)'s such that \( b_n = 1 \) and \( b_1, \ldots, b_{n-1} \) can be chosen either 0 or 1, there are \( 2^{n-1} \) such choices.

(ii) To show \( A_1, A_2, \ldots, A_n \) are mutually independent, we need to show for any subcollection \( \{A_{i_1}, A_{i_2}, \ldots, A_{i_k}\} \subseteq \{A_1, A_2, \ldots, A_n\} \)
\[ \mathbb{P}[\bigcap_{j=1}^{k} A_{i_j}] = \prod_{j=1}^{k} \mathbb{P}[A_{i_j}] \]

We log \( \log \) let \( i_1 < i_2 < \ldots < i_k \)

Then \[ \mathbb{P}[A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}] = \mathbb{P}[\{ \omega \in \Omega : \omega_{i_1} = 1, \omega_{i_2} = 1, \ldots, \omega_{i_k} = 1 \}] \]

Note: \[ \mathbb{P}[A_{i_j}] = \frac{1}{2} \]
\[ \forall j = 1, \ldots, k \]
from part (i)
\[
\begin{align*}
&= \sum_{b_j \in \{0,1\} \setminus \{i_{k-1}, \ldots, i_k\}} P \left[ E_{i_k} \left( b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_{i_k} \right) \right] \\
&= 2^{-k} \left( i_k - k \right) - i_k \\
&= 2^{-k} \\
&= \prod_{j=1}^{k} P \left[ A_{i_j} \right] \\
&= \prod_{j=1}^{k} \left[ \frac{1}{2} \right] \\
&= \frac{1}{2^k} \\
\end{align*}
\]

Since we need to sum over all such \( E_{i_k} \) that \( b_{i_1} = b_{i_2} = \ldots = b_{i_k} = 1 \) and all other \( b_j \) can be chosen either '0' or '1', clearly there are \( 2^{i_k - k} \) no. of choice.

\( \therefore \) \( A_1, \ldots, A_n \) are mutually independent.

\( \text{iii) } \]
\[
\begin{align*}
\Pr \left[ A_n \right] &= \frac{1}{2} \\
\implies \sum_{n=1}^{\infty} \Pr \left[ A_n \right] &= \sum_{n=1}^{\infty} \frac{1}{2} = \infty \\
\text{From (ii) } A_n \text{; } n = 0, 1, 2, \ldots \text{ are mutually independent.} \\
\text{From BC Lemma II } \]
\[
\Pr \left[ A_n \mid 0 \right] = 1
\]

It certainly does match the intuition! \( A_n = \{ w \in \mathbb{N}, w_n = 1 \} \) is the event that we get '1' in an infinite coin toss (with \{0,1\} faces) \( n \)-th trial. Therefore \( [A_n \mid 0] \) denotes the event that we get '1' infinitely often in the series of tosses, i.e., \( \forall k \in \mathbb{N}, \exists \ n \geq k \) such that we get '1' in the \( n \)-th toss, however large \( k \) may be. This goes well with our intuition that this is a certain occurrence.
Alternative Way:

\[
[An \text{ i.o.}] = [An \text{ does not occur at all}]^c = [\{\text{getting an all '0' sequence}\}]^c
\]

\[
= 1 - P[\{\text{getting an all '0' sequence}\}]
\]

\[
= 1 - \frac{1}{2} \times \frac{1}{2} \times \ldots \infty \approx 1
\]

Intuitively, the probability of getting all '0' in an infinite coin toss is zero as well, so \( P[An \text{ i.o.}] \) must be unity.
10. \[
\{F_i, \ i \in I\} \text{ is a family of } \Omega \text{-field on } \mathbb{R} \\
\{E \in \mathcal{P}(\mathbb{R}) : E \in F_i, \ \forall i \in I\}
\]
\[
= \bigcap_{i \in I} F_i
\]
\[
\bigcap_{i \in I} F_i \text{ is a } \Omega \text{-field.}
\]
Proof: • Since \( F_i, i \in I \) are \( \Omega \) fields,
\[
\forall E \in F_i, \ \forall i \in I
\]
\[
\Rightarrow \forall E \in \bigcap_{i \in I} F_i
\]
Similarly \( \exists \in \bigcap_{i \in I} F_i
\)

• Let \( E \in \bigcap_{i \in I} F_i \Rightarrow E \in F_i,\ \forall i \in I
\]
\[
= \bigcap_{i \in I} E \in F_i,\ \forall i \in I
\]
\[
\text{Since } F_i, i \in I \text{ are } \Omega \text{-field,}
\]
\[
\Rightarrow E \in \bigcap_{i \in I} F_i
\]

• Let \( \{E_j, \ j \in J\} \subseteq \bigcap_{i \in I} F_i \) where \(|J| < \aleph_0
\)
Now that means
\[
\{E_j, \ j \in J\} \subseteq F_i, \ \forall i \in I
\]
\[
\Rightarrow \bigcup_{j \in J} E_j \in F_i, \ \forall i \in I \quad \lbrack \text{Each } F_i \text{ is } \Omega \text{-field}\rbrack
\]
\[
\Rightarrow \bigcup_{j \in J} E_j \in \bigcap_{i \in I} F_i
\]