Please work out the **ten** (10) problems stated below; **show** work and **explain** reasoning. When not specified, an underlying probability triple \((\Omega, \mathcal{F}, \mathbb{P})\) is always assumed. Throughout \(p\) and \(q\) are positive integers.

1. Let \(X\) and \(Y\) be two i.i.d. zero-mean Gaussian rv with variance \(\sigma^2 > 0\). Use conditioning arguments to find the probability distribution function and the probability density function of the rv \(R\) given by
\[
R = \sqrt{X^2 + Y^2}.
\]
[HINT: An efficient way to get the answer is to start with \(R^2\)]. This probability distribution is known as the Rayleigh distribution; it occurs in the context of digital communications where it is used to model fading.

2. Consider the rvs \(X_1, \ldots, X_n : \Omega \to \mathbb{R}\) with characteristic functions \(\varphi_1, \ldots, \varphi_n : \mathbb{R} \to \mathbb{C}\).

2.a. Construct a rv \(Y : \Omega \to \mathbb{R}\) whose characteristic function \(\varphi_Y : \mathbb{R} \to \mathbb{C}\) is given by
\[
\varphi_Y(t) = \sum_{i=1}^{n} a_i \varphi_i(t), \quad t \in \mathbb{R}
\]
with
\[
0 < a_i < 1, \quad i = 1, \ldots, n \quad \text{and} \quad a_1 + \ldots + a_n = 1.
\]

2.b. Construct a rv \(Z : \Omega \to \mathbb{R}\) (through its probability distribution function) whose characteristic function \(\varphi_Z : \mathbb{R} \to \mathbb{C}\) is given by
\[
\varphi_Z(t) = \int_{0}^{\infty} \varphi_1(ut)e^{-u}du, \quad t \in \mathbb{R}.
\]
3. Consider two independent zero-mean Gaussian rvs $X_1$ and $X_2$ with $\sigma^2_k = \text{Var}[X_k] > 0$ for $k = 1, 2$. The rv $X$ is defined by

$$X = \frac{X_1 \cdot X_2}{\sqrt{X_1^2 + X_2^2}} 1 \left[ X_1^2 + X_2^2 > 0 \right].$$

3.a. Show that the rv $X$ has a symmetric distribution in the sense that $-X$ has the same distribution as $X$.

3.b. Use Part a to argue that knowing the probability distribution of $X$ is equivalent to knowing the probability distribution of $X^2$. In particular how is the probability distribution of the former related to that of the latter?

3.c. Use Part b to find the probability distribution of the rv $X$.

4. The following arises in classical Statistics: Let $X_1, \ldots, X_n$ denote $n$ i.i.d. Gaussian rvs, each with mean $\mu$ and variance $\sigma^2 > 0$. Define the rvs $\bar{X}$ and $Z_1, \ldots, Z_n$ by

$$\bar{X} = \frac{1}{n} \sum_{k=1}^n X_k$$

and

$$Z_k = X_k - \bar{X}, \quad k = 1, 2, \ldots, n.$$

4.a. Compute the joint characteristic function of the $n + 1$ rvs $Z_1, \ldots, Z_n$ and $\bar{X}$.

4.b. Use Part a to establish the independence of the rvs $\bar{X}$ and $S^2$ where

$$S^2 = \frac{1}{n-1} \sum_{k=1}^n (X_k - \bar{X})^2.$$

5. The rvs $X_1, \ldots, X_n$ are jointly Gaussian, e.g., with $X = (X_1, \ldots, X_n)'$, we have

$$X \sim \text{N}(\mu, R)$$

for some vector $\mu$ in $\mathbb{R}^n$ and $n \times n$ covariance matrix $R$. With $a$ and $b$ elements in $\mathbb{R}^n$, define the $\mathbb{R}$-valued rvs $A$ and $B$ by

$$A := a'X = \sum_{k=1}^n a_k X_k \quad \text{and} \quad B := b'X = \sum_{k=1}^n b_k X_k.$$

5.a. Compute the characteristic function of the $\mathbb{R}^2$-valued rv $(A, B)'$, namely

$$\varphi(s, t) = \mathbb{E} \left[ e^{i(sA + tB)} \right], \quad s, t \in \mathbb{R}. $$
Carefully explain your calculations!

5.b. With the help of your answer in Part a derive a necessary and sufficient condition on the parameters $\mu$, $a$, $b$ and $R$ for the rvs $A$ and $B$ to be independent. Carefully explain your calculations!

5.c. What form does this condition take when the rvs $X_1, \ldots, X_n$ are i.i.d. Gaussian rvs, say $X \sim N(\mu, \sigma^2 I_n)$ with $\sigma^2 > 0$?

6. Consider the bivariate Gaussian rv $(X, Y)'$ with probability density function $f_{X,Y} : \mathbb{R}^2 \to \mathbb{R}_+$ given by

$$f_{X,Y}(x, y) = \frac{1}{2\pi} e^{-\frac{1}{2}(2x^2+y^2+2xy-22x-14y+65)}, \quad (x, y) \in \mathbb{R}^2.$$  

Evaluate the quantities $E[X]$, $E[Y]$, $\text{Var}[X]$, $\text{Var}[Y]$ and $\text{Cov}[X, Y]$.

7. Problem 3.1 (BH).

8. Problem 3.4 (BH).

9. Let $\xi, \eta : \Omega \to \mathbb{R}$ be independent rvs, each of which is distributed according to a standard Gaussian distribution. Define the rv $(\xi^*, \eta^*) : \Omega \to \mathbb{R}^2$ given by

$$\begin{pmatrix} \xi^* \\ \eta^* \end{pmatrix} = \begin{cases} \begin{pmatrix} \xi \\ |\eta| \end{pmatrix} & \text{if } \xi \geq 0 \\ \begin{pmatrix} \xi \\ -|\eta| \end{pmatrix} & \text{if } \xi < 0 \end{cases}.$$  

Show that rvs $\xi^*$ and $\eta^*$ are standard Gaussian rvs but that the rv $(\xi^*, \eta^*) : \Omega \to \mathbb{R}^2$ is not Gaussian. Contrast with the statement: The rv $(\xi, \eta) : \Omega \to \mathbb{R}^2$ is a jointly Gaussian rv $N(0_2, I_2)$ with $0_2 = (0, 0)'$ and $I_2$ is the identity on $\mathbb{R}^2$. What explain the difference?

10. The rvs $X_1, \ldots, X_k$ are defined on the probability triple $(\Omega, \mathcal{F}, P)$. For each $\ell = 1, \ldots, k$, the rv $X_\ell : \Omega \to \mathbb{R}^{p_\ell}$ is a Gaussian rv $N(\mu_\ell, R_\ell)$ where $\mu_\ell$ is an element of $\mathbb{R}^{p_\ell}$ for some positive integer $p_\ell$ and $R_\ell$ is a symmetric positive semi-definite $p_\ell \times p_\ell$ matrix. With $p = p_1 + \ldots + p_k$ consider the rv $X : \Omega \to \mathbb{R}^p$ given by

$$X \equiv \begin{pmatrix} X_1 \\ \vdots \\ X_k \end{pmatrix}.$$
10.a. Show that $X$ is a Gaussian rv $N(\mu, R)$ if the rvs $X_1, \ldots, X_k$ are mutually independent rvs. Construct the corresponding parameters $\mu$ ad $R$ explicitly.

10.b. Show that if $X$ is a Gaussian rv $N(\mu, R)$, then the rvs $X_1, \ldots, X_k$ are mutually independent rvs if and only if they are uncorrelated.

10.c. TRUE/FALSE: If the rvs $X_1, \ldots, X_k$ are uncorrelated, then the $X$ is a Gaussian rv $N(\mu, R)$. Either prove the statement or give a counterexample!