1. Since $R$ is symmetric we need only show that it is positive semi-definite. We write any element $v$ in $\mathbb{R}^{n+1}$ as

$$v = \begin{bmatrix} x \\ y \end{bmatrix}$$

with $x$ in $\mathbb{R}$ and $y = (y_1, \ldots, y_n)'$ in $\mathbb{R}^n$. With this notation we get

$$Rv = \begin{bmatrix}
\sigma^2 x + \sum_{k=1}^{n} \rho_k y_k \\
\rho_1 x + \sigma_1^2 y_1 \\
\rho_2 x + \sigma_2^2 y_2 \\
\vdots \\
\rho_k x + \sigma_k^2 y_k \\
\vdots \\
\rho_n x + \sigma_n^2 y_n
\end{bmatrix}$$

so that

$$v'Rv = \left(\sigma^2 x + \sum_{k=1}^{n} \rho_k y_k\right) x + \sum_{k=1}^{n} y_k \left(\rho_k x + \sigma_k^2 y_k\right)$$

$$= \sigma^2 x^2 + 2 \sum_{k=1}^{n} \rho_k xy_k + \sum_{k=1}^{n} \sigma_k^2 y_k^2$$

$$= \sigma^2 x^2 + \sum_{k=1}^{n} \left(\sigma_k^2 y_k^2 + 2 \rho_k xy_k\right)$$

$$= \sigma^2 x^2 + \sum_{k=1}^{n} \left(\sigma_k^2 y_k^2 + 2 \frac{\rho_k}{\sigma_k} x (\sigma_k y_k)\right)$$

$$= \sigma^2 x^2 + \sum_{k=1}^{n} \left(\frac{\rho_k}{\sigma_k} x + \sigma_k y_k\right)^2 - \sum_{k=1}^{n} \left(\frac{\rho_k}{\sigma_k} x\right)^2$$

$$= \left(\sigma^2 - \sum_{k=1}^{n} \left(\frac{\rho_k}{\sigma_k}\right)^2\right) x^2 + \sum_{k=1}^{n} \left(\frac{\rho_k}{\sigma_k} x + \sigma_k y_k\right)^2$$

(1.1)

It is now plain that $R$ is positive semi-definite, i.e., $v'Rv \geq 0$ for all $v$ in $\mathbb{R}^{n+1}$, if and only if

$$\sigma^2 - \sum_{k=1}^{n} \left(\frac{\rho_k}{\sigma_k}\right)^2 \geq 0.$$
This condition is now enforced on the parameters entering $\mathbf{R}$.
Many of you used a criterion for positive semi-definiteness that involves the leading principal minors. Unfortunately, this condition is necessary and sufficient for positive definiteness, but only necessary for semi-positive definiteness.

1.b. The $(n+1)$-dimensional random vector $(X, Y_1, \ldots, Y_n)'$ is assumed to be normally distributed $\mathcal{N}((0, 0, \ldots, 0)', \mathbf{R})$. The existence of a probability density function is equivalent to $\mathbf{R}$ being invertible (i.e., positive definite), and this occurs if and only if

$$\sigma^2 - \sum_{k=1}^{n} \left( \frac{\rho_k}{\sigma_k} \right)^2 > 0.$$

Indeed, $\mathbf{v}'\mathbf{R}\mathbf{v} = 0$ occurs if and only if

$$\left( \sigma^2 - \sum_{k=1}^{n} \left( \frac{\rho_k}{\sigma_k} \right)^2 \right) x^2 = 0$$

and

$$\left( \frac{\rho_k}{\sigma_k} x + \sigma_k y_k \right)^2 = 0, \quad k = 1, \ldots, n$$

If the condition $\mathbf{v}'\mathbf{R}\mathbf{v} = 0$ must imply $0_{n+1}$, then we necessarily have

$$\left( \sigma^2 - \sum_{k=1}^{n} \left( \frac{\rho_k}{\sigma_k} \right)^2 \right) \neq 0$$

and the announced condition follows!

1.c. By Part 1.b, the probability distribution function of the $(n+1)$-dimensional random vector $(X, Y_1, \ldots, Y_n)'$ will not admit a probability density function if and only if

$$\sigma^2 - \sum_{k=1}^{n} \left( \frac{\rho_k}{\sigma_k} \right)^2 = 0,$$

in which case

$$\mathbf{v}'\mathbf{R}\mathbf{v} = \sum_{k=1}^{n} \left( \frac{\rho_k}{\sigma_k} x + \sigma_k y_k \right)^2, \quad \mathbf{v} \in \mathbb{R}^{n+1}.$$

Therefore, $\mathbf{v}'\mathbf{R}\mathbf{v} = 0$ if and only if

$$\frac{\rho_k}{\sigma_k} x + \sigma_k y_k = 0, \quad k = 1, 2, \ldots, n$$

i.e.,

$$y_k = -\left( \frac{\rho_k}{\sigma_k^2} \right) x, \quad k = 1, 2, \ldots, n$$

This suggests introducing the linear subspace $K$ of $\mathbb{R}^{n+1}$ given by

$$K = \{ \mathbf{v} \in \mathbb{R}^{n+1} : \mathbf{v} = x\mathbf{a}, \; x \in \mathbb{R} \} = \mathbb{R}\mathbf{a}$$
where
\[ a = \left( 1, -\left( \frac{\rho_1}{\sigma_1^2} \right), \ldots, -\left( \frac{\rho_n}{\sigma_n^2} \right) \right) \neq 0_{n+1}. \]

Note that \( \dim(K) = 1 \).

It now follows in the usual manner that
\[
\Var [xX + y'Y] = v'Rv = \sum_{k=1}^{n} \left( \frac{\rho_k x + \sigma_k y_k}{\sigma_k} \right)^2 = 0, \quad v \in K
\]
in which case
\[ xX + y'Y = 0 \ a.s. \]
whenever \( v = (x, y')' \) is an element of \( K \). Thus,
\[
\mathbb{P} [(X, Y_1, \ldots, Y_n)' \in H] = 1
\]
where
\[
H = K^\perp = \{ v \in \mathbb{R}^{n+1} : v'a = 0 \}
= \left\{ (x, y')' \in \mathbb{R}^{n+1} : x = \sum_{k=1}^{n} \frac{\rho_k}{\sigma_k} y_k \right\}
\]
and \( \dim(K) = n \).

2. 

2.a. For each \( t \) in \( \mathbb{R} \), note that
\[
\mathbb{E} \left[ e^{itX} \right] = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} e^{ikt} = e^{-\lambda(1-e^{it})}
\]
by standard calculations, with a similar expression for \( \mathbb{E} \left[ e^{itY} \right] \). By the independence of the rvs \( X \) and \( Y \), hence of the rvs \( e^{itX} \) and \( e^{-itY} \), we get
\[
\mathbb{E} \left[ e^{it(X-Y)} \right] = \mathbb{E} \left[ e^{itX} \right] \cdot \mathbb{E} \left[ e^{-itY} \right] = e^{-\lambda(1-e^{it})} \cdot e^{-\lambda(1-e^{-it})}
= e^{-2\lambda(1-\cos t)}
\]
as we recall the identity
\[
\cos t = \frac{e^{it} + e^{-it}}{2}.
\]
2.b. Fix $n = 1, 2, \ldots$ and $t$ in $\mathbb{R}$. For each $k = 1, 2, \ldots, n$, we find
\[
\mathbb{E} \left[ e^{it \frac{X_{n,k}}{\sqrt{n}}} \right] = \frac{1}{2n} \left( e^{it} + e^{-it} \right) + 1 - \frac{1}{n}
\]
\[
= \frac{1}{n} \cos t + 1 - \frac{1}{n}
\]
\[
= \left( 1 - \frac{1}{n} \right) \left( 1 - \cos t \right)
\]
(1.5)

Using the independence of the rvs $X_{n,1}, \ldots, X_{n,n}$ we get
\[
\mathbb{E} \left[ e^{it \frac{S_n}{\sqrt{n}}} \right] = \prod_{k=1}^{n} \mathbb{E} \left[ e^{it \frac{X_{n,k}}{\sqrt{n}}} \right]
\]
\[
= \left( 1 - \frac{1}{n} \right) \left( 1 - \cos t \right)^n
\]
(1.6)

2.c. It is now plain that
\[
\lim_{n \to \infty} \mathbb{E} \left[ e^{it \frac{S_n}{\sqrt{n}}} \right] = \lim_{n \to \infty} \left( 1 - \frac{1}{n} \right) \left( 1 - \cos t \right)^n = e^{-(1-\cos t)}
\]
for each $t$ in $\mathbb{R}$, thus $\frac{S_n}{\sqrt{n}} \Rightarrow_n L$ where $L$ is distributed like the difference of two independent Poisson rvs with parameter $\lambda = \frac{1}{2}$. This is an immediate consequence of Part 2.a.

3. There are several different ways to show that
\[
\lim_{n \to \infty} \frac{X_n}{\sqrt{n}} = 0 \quad a.s.
\]
(1.7)

for each $p \geq 1$.

**First approach** – The rvs $\{X, X_n, n = 1, 2, \ldots\}$ are i.i.d. rvs, each exponentially distributed with unit parameter. Thus,
\[
\mathbb{E} \left[ X_k \right] < \infty, \quad k = 1, 2, \ldots
\]
(1.8)

so that $\mathbb{E} [X^p] < \infty$ for each $p \geq 1$. Just apply (1.8) with $k(p) = \lceil p \rceil$ and use the fact that $\mathbb{E} [X^p]$ is necessarily finite since $p < \lceil p \rceil$.

The rvs $\{(X_n)^p, n = 1, 2, \ldots\}$ are still independent and identically distributed, and by the Strong Law of Large Numbers applied to this sequence of rvs we conclude that
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (X_k)^p = \mathbb{E} [X^p] \quad a.s.
\]
whence
\[
\lim_{n \to \infty} \frac{(X_n)^p}{n} = 0 \quad a.s.
\]

by standard arguments. This establishes (1.7).
Second approach – For every \( \varepsilon > 0 \), note that
\[
\mathbb{P} \left[ \frac{X_n}{\sqrt{n}} > \varepsilon \right] = \mathbb{P} \left[ X_n > \varepsilon \sqrt{n} \right] = e^{-\varepsilon \sqrt{n}}
\]
for each \( n = 1, 2, \ldots \). Thus,
\[
\sum_{n=1}^{\infty} \mathbb{P} \left[ \frac{X_n}{\sqrt{n}} > \varepsilon \right] = \sum_{n=1}^{\infty} e^{-\varepsilon \sqrt{n}} < \infty
\]
and the a.s. convergence (1.7) follows.

3.b. For each \( n = 1, 2, \ldots \), elementary calculations yield
\[
R_n = \sqrt{T_{n+1}} - \sqrt{T_n} = \frac{\sqrt{T_{n+1}} - \sqrt{T_n} \cdot (\sqrt{T_{n+1}} + \sqrt{T_n})}{\sqrt{T_{n+1}} + \sqrt{T_n}} = \frac{T_{n+1} - T_n}{\sqrt{T_{n+1}} + \sqrt{T_n}} = \frac{X_{n+1}}{\sqrt{T_{n+1}} + \sqrt{T_n}} = \frac{X_{n+1}}{\sqrt{T_{n+1}T_n}}.
\] (1.9)

By the Strong of Large Numbers, we have
\[
\lim_{n \to \infty} \frac{T_{n+1}}{n+1} = 1 \text{ a.s.} \quad \text{and} \quad \lim_{n \to \infty} \frac{T_n}{n+1} = 1 \text{ a.s.}
\]
Using these facts and the convergence (1.7) (for \( p = 2 \)) we find
\[
\lim_{n \to \infty} R_n = 0 \text{ a.s.}
\]

4. 

4.a. Here the Orthogonality Principle reads
\[
\mathbb{E}[(X - Z^*) Z] = 0, \quad Z \in V. \tag{1.10}
\]
However, any element \( Z \) of \( V \) is of the form
\[
Z = \sum_{k=1}^{n} a_k Y_k + b
\]
with arbitrary \( a_1, \ldots, a_n \) and \( b \) in \( \mathbb{R} \). Obviously, the rvs \( Z = Y_1, \ldots, Z = Y_n \) and \( Z = 1 \) are in \( V \). Using them in (1.10) we get
\[
\mathbb{E}[(X - Z^*) 1] = 0 \quad \text{and} \quad \mathbb{E}[(X - Z^*) Y_k] = 0, \quad k = 1, \ldots, n.
\]
Thus, $Z^\star$ satisfies
\[
\mathbb{E}[X] = \mathbb{E}[Z^\star] \quad \text{and} \quad \mathbb{E}[(X - \mathbb{E}[X] + Z^\star)Y_k] = 0, \quad k = 1, \ldots, n
\]
and this is equivalent to
\[
\mathbb{E}[X] = \mathbb{E}[Z^\star] \quad \text{and} \quad \text{Cov}[X - Z^\star, Y_k] = 0, \quad k = 1, \ldots, n. \quad (1.11)
\]
By linearity it is elementary to see that (1.10) and (1.11) are indeed equivalent.

4.b. Recall that $Z^\star$ is of the form
\[
Z^\star = \sum_{\ell=1}^n a^*_\ell Y_\ell + b^*
\]
with $a^*_1, \ldots, a^*_n$ and $b^*$ in $\mathbb{R}$ such that (1.11) holds. In particular,
\[
\mathbb{E}[X] = \sum_{\ell=1}^n a^*_\ell \mathbb{E}[Y_\ell] + b^*
\]
and for each $k = 1, 2, \ldots, n$, we find
\[
\text{Cov}[X - Z^\star, Y_k] = \text{Cov}[X - \left(\sum_{\ell=1}^n a^*_\ell Y_\ell + b^*\right), Y_k] = \text{Cov}[X, Y_k] - \sum_{\ell=1}^n a^*_\ell \text{Cov}[Y_\ell, Y_k] = \text{Cov}[X, Y_k] - a^*_k \text{Var}[Y_k] = \rho_k - a^*_k \sigma^2_k \quad (1.12)
\]
whence
\[
a^*_k = \frac{\rho_k}{\sigma^2_k}.
\]
It is now plain that
\[
Z^\star = \sum_{\ell=1}^n a^*_\ell Y_\ell + b^*
\]
\[
= \sum_{\ell=1}^n a^*_\ell Y_\ell + \left(\mathbb{E}[X] - \sum_{\ell=1}^n a^*_\ell \mathbb{E}[Y_\ell]\right)
\]
\[
= \mathbb{E}[X] + \sum_{\ell=1}^n a^*_\ell (Y_\ell - \mathbb{E}[Y_\ell])
\]
\[
= \mathbb{E}[X] + \sum_{\ell=1}^n \frac{\rho_\ell}{\sigma^2_\ell} (Y_\ell - \mathbb{E}[Y_\ell]). \quad (1.13)
\]
4.c. Next, we note that

\[ X - Z^* = X - \mathbb{E}[X] - \sum_{\ell=1}^{n} \frac{\rho_{\ell}}{\sigma_{\ell}^2} (Y_{\ell} - \mathbb{E}[Y_{\ell}]) \]

so that

\[
\text{Var}[X - Z^*] = \mathbb{E}[(X - Z^*)^2] = \mathbb{E} \left[ \left( X - \mathbb{E}[X] - \sum_{\ell=1}^{n} \frac{\rho_{\ell}}{\sigma_{\ell}^2} (Y_{\ell} - \mathbb{E}[Y_{\ell}]) \right)^2 \right] = \text{var}[X] + \sum_{\ell=1}^{n} \left( \frac{\rho_{\ell}}{\sigma_{\ell}} \right)^2 \sigma_{\ell}^2 - 2 \sum_{\ell=1}^{n} \frac{\rho_{\ell}}{\sigma_{\ell}^2} \text{Cov}[X, Y_{\ell}] = \text{var}[X] + \sum_{\ell=1}^{n} \left( \frac{\rho_{\ell}}{\sigma_{\ell}} \right)^2 \sigma_{\ell}^2 - 2 \sum_{\ell=1}^{n} \left( \frac{\rho_{\ell}}{\sigma_{\ell}} \right)^2 \sigma_{\ell}^2 - 2 \sum_{\ell=1}^{n} \left( \frac{\rho_{\ell}}{\sigma_{\ell}} \right)^2 \sigma_{\ell}^2 = \text{var}[X] - \sum_{\ell=1}^{n} \left( \frac{\rho_{\ell}}{\sigma_{\ell}} \right)^2 \sigma_{\ell}^2 \]

(1.14)