These notes are a collection which I think will be useful as a review of what you may already know. No proofs are given, though some relationships may be shown in more detail than others, especially as these apply to the machines lab.

The Fourier transform is defined on the imaginary axis while the Laplace transform has a wider region of convergence (ROC) extending over the s-plane. \( s = \sigma + j\omega \). In sinusoidal steady state (SSS) analysis, transients are neglected. It can be shown that this is similar to letting \( \sigma = 0 \), or \( s = j\omega \). We shall use examples below to show how this is applied to power circuits.

To begin with a list of a few useful Laplace transforms properties is shown below:

If \( x(t) \leftrightarrow L \rightarrow X(s) \), with \( ROC = R \), then
\[
\frac{dx(t)}{dt} \leftrightarrow L \rightarrow sX(s), \text{ with } ROC \text{ containing } R, \text{ and }
\]
\[
\int_{-\infty}^{t_0} x(\tau)d\tau \leftrightarrow L \rightarrow \frac{1}{s} X(s), \text{ with } ROC \text{ containing } R \cap \{ \Re\{s\} > 0 \}.
\]

In the case of elementary power circuits, we should have little concern about the \( ROC \), hence, it is not mentioned below. This is equivalent to the statement: differentiation in the time domain is equivalent to multiplication in the \( s \)-domain and integration in the time domain is equivalent to division by \( s \) in the \( s \)-domain. In the SSS, differentiation in the time domain is equivalent to multiplication by \( j\omega \) in the frequency domain, while integration in the time domain is equivalent to division by \( j\omega \) in the frequency domain.

Since we are very much interested in the sinusoidal steady state (SSS), we will examine the Laplace transform of these sinusoidal functions. First, the relation between the \( \sin(.) \) and \( \cos(.) \) is given below, followed by the Laplace transforms as a transform pair.
\[
\frac{d}{dt}\sin(\omega t) = \omega \cos(\omega t) \quad \sin(\omega t) = \omega \int \cos(\omega \tau) d\tau
\]
\[
\begin{align*}
\sin(\omega t) & \leftrightarrow L \rightarrow \frac{\omega}{s^2 + \omega^2} \\
\cos(\omega t) & \leftrightarrow L \rightarrow \frac{s}{s^2 + \omega^2}
\end{align*}
\]

N.B. From now on (and in the equations above) we assume that all time functions are zero for negative time, for example, by \( \sin(\omega t) \) we mean \( \sin(\omega t)u(t) \) where \( u(t) \) is the unit step function.

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1 In making most of these notes, I have borrowed freely from Chapter 2 of the book *Power System Analysis* by Arthur R. Bergen, published by Prentice-Hall, 1986. I have also used material from Chapter 2 of the book *Power System Analysis* by Hadi Saadat, published by McGraw-Hill, 1999. The reader is referred to these books for further details. You can also use other books at the library.
Now we turn our attention to phasors. Let \( v(t) = A \cos(\omega t + \theta) \). This can be represented as: \( v(t) = A R e^{j\omega t + \theta} \). This is easily seen as the projection of a rotating phasor \( V \) (whose amplitude is \( A \) and whose angle is \( \omega t + \theta \)) onto the x-axis in the x-y plane. The picture shown below is taken at \( t = 0 \). As \( t \) increases, the phasor \( V \) rotates CCW at an angular speed of \( \omega \) rad/s.

If we allow the axes to rotate CCW at the same rate of \( \omega \) rad/s as the phasor \( V \), then the phasor \( V \) becomes the stationary vector \( V \) and the figure on the right becomes independent of time. This is how all sinusoidal quantities in a power system are represented. In this case, the quantity \( v(t) = A \cos(\omega t + \theta) \) is represented as the vector \( V \) and is denoted as \( A \angle \theta \). Other notation includes \( Ae^{j\theta} \), \( |V| \angle \theta \), and \( A(\cos\theta + j\sin\theta) \).

The following trigonometric identities are useful in the next step:

\[
\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y \quad \text{and} \quad \sin(x \pm y) = \sin x \cos y \pm \cos x \sin y
\]

Thus we have: \( \sin(\omega t + \theta + 90^\circ) = \cos(\omega t + \theta) \). Hence we can use the \( \cos \) and \( \sin \) reference directions shown on the figure to the right to interpret a vector as a phasor using either a \( \sin \) or \( \cos \) function. As another example, consider the phasor \( W = 10 \cos(\omega t - 25^\circ) \) which is the same as the phasor \( W = 10 \sin(\omega t + 65^\circ) \). This is shown in the figure below.

Thus the positive x-axis is the \( \cos \) reference while the negative y-axis is the \( \sin \) reference. This is very handy in converting phasors from a \( \cos \) reference and vise-versa.

It is further noted that addition of \( \sin \)es and \( \cos \)ines can be done by simply adding their respective phasors (vector sum). This greatly simplifies the addition and subtraction of sinusoidal forms (at the same frequency).

In the notation above, a differentiation is a rotation of \( 90^\circ \) CCW and an amplitude factor of \( \omega \). An integration produces a \( 90^\circ \) CW rotation and an amplitude factor of \( \frac{1}{\omega} \). For example, the vector \( W = 10 \sin(\omega t + 65^\circ) \) can be differentiated wrt time, the result is:
\[
\frac{dW}{dt} = 10\omega \cos(\omega t + 65^\circ) = D.
\]
This is clearly the same vector as \( W \) rotated 90° CCW and multiplied by \( \omega \). If we integrate, the result is
\[
W(t)dt = -\frac{10}{\omega} \cos(\omega t + 65^\circ) = Y
\]
which is the same as the vector \( W \) divided by \( \omega \) and rotated 90° CW. This is easy to see in another way: If we look at the Laplace transform of a function and its time derivative, we see that differentiation in the time domain is like multiplication by \( s \) in the \( s \)-plane. In the SSS, this is like vector multiplication by \( j\omega \) (rotate 90° CCW and multiply by \( \omega \)). Integration in the time domain is equivalent to division by \( s \) in the Laplace transform. In the SSS, this is like dividing by the vector \( j\omega \) (rotate 90° CW and divide by \( \omega \)).

These vectors \( (W, D, \text{and } Y) \) are shown in the figure below where we assume \( \omega = 2 \) for illustrative purposes.

Next we derive an expression for the complex power \( S \) for a two-port \( N \). This is an elementary network as shown below. According to convention, if the power is positive then it is being “absorbed” in the network, if negative, then the network is “providing” the power to the outside. Let \( v(t) = V_m \cos(\omega t + \theta_v) \) and \( i(t) = I_m \cos(\omega t + \theta_i) \). The power is thus expressed as:
\[
p(t) = V_m I_m \cos(\omega t + \theta_v) i(t) \cos(\omega t + \theta_i) \quad \text{or}, \quad p(t) = \frac{1}{2} V_m I_m \left[ \cos \theta_v - \theta_i + \cos 2\omega t + \theta_v + \theta_i \right].
\]
As can be seen, the power has two components, the first is independent of time (constant) while the other is a sinusoid at twice line frequency. The twice line frequency is easy to predict as follows. Since \( p(t) \) is the product of \( v(t) \) and \( i(t) \), then every time either \( v(t) \) or \( i(t) \) crosses zero, \( p(t) \) must also cross zero. Thus the frequency of \( p(t) \) is the sum of frequencies of \( v(t) \) and \( i(t) \), or \( 2\omega t \). In power circuits we are more interested in the average power than the instantaneous power, thus we define \( P \) as the average value of \( p(t) \):
\[
P = \text{avg} \ p(t) = \frac{1}{2} V_m I_m \left[ \cos \theta_v - \theta_i \right].
\]
If we let \( \phi = \theta_v - \theta_i \), then \( P = \frac{1}{2} V_m I_m \cos \phi \). If we let \( V = V_{rms} = V_m / \sqrt{2} \) and \( I = I_{rms} = I_m / \sqrt{2} \), then \( P = VI \cos \phi \). \( \cos \phi \) is called the \textit{power factor} (PF or p.f. for short) while \( \phi \) is called the \textit{power factor angle}. For example, for a resistor, \( \phi = 0 \) and the average power is given by \( P = VI \). For a capacitor, \( \phi = -90^\circ \), thus \( P = 0 \). For an inductor, \( \phi = 90^\circ \) and \( P = 0 \). For an inductive RL circuit, \(-90^\circ \leq \phi \leq 0^\circ \). For a capacitive RC circuit, \( 0^\circ \leq \phi \leq 90^\circ \). Since the cosine is an even function, the power factor does not give us information as to the sign of \( \phi \). Thus if the power factor is 0.5, we could not say if the angle is +60° or −60°. To distinguish these two cases we say in the case of the
capacitive (RC) network the power factor is “0.5 leading” (the current leads the voltage). For an inductive (RL) network, the power factor would be “0.5 lagging” (again, current lagging voltage). Note that the angle \( \phi \) is defined as the angle by which the voltage leads the current, which is the exact opposite convention. This will manifest itself in a conjugation of the current when computing the complex power as we shall see below. The Matlab program below illustrates this for a capacitive network:

```matlab
clear;clf;t=linspace(0,.03);%generate 100 points for time, 0 to 30 ms
v=1.1*cos(377*t);           %let the voltage be 1.1 V at 377 rad/s
i=2*cos(377*t+60*pi/180);   %let the current be 2 A, 60 deg lead
p=(v.*i);                   %find the instantaneous power
Pa=1.1*2*cos(60*pi/180)/2;  %average power
Pav=Pa.*ones(size(t)); z=zeros(size(t));
plot(t,i,'r-',t,v,'g--',t,p,'b-.', t,Pav,'k-',t,z,'k-');
grid; xlabel('time');title('V, I, p(t) and Pavg);
```

The curves in the figure above represent the voltage and current (at the same frequency), instantaneous power (at twice the frequency), and the average power (constant).

Various expressions for the power in a resistor are: \( P = VI = V^2 / R = I^2 R \). Here, of course, both \( V \) and \( I \) are rms values. To distinguish between rms vectors and rms values, we should use magnitude signs (which we often omit for brevity at the expense of some effort to understand this implication). Thus, the above expressions are more pre-
cissely written as: \( P = \left| V \right| \left| I \right| = \left| V \right|^2 / R = \left| I \right|^2 R \) where rms values are used. Going back to the original expression for the general value of power on the network \( N \) we have:

\[ P = \left| V \right| \cos \theta \cos \theta, = \mathfrak{R} \left| e^{\left( \theta \right)} \right| \left| I \right|^2 = \mathfrak{R} \left| V \right| I . \]

Note the conjugation of the current as mentioned earlier, top of page 3. Thus we now define the complex power as \( S = VI^*. \) Clearly \( P = \mathfrak{R} \left| S \right| \). Let \( Q \) be the imaginary part of \( S \) then

\[ S = VI^* = \left| V \right| \left| I \right| e^{\left( \theta \right)} = P + jQ , \]

where \( V \) and \( I \) are rms values. More will be said about \( Q \) below. From the definition above, \( Q = \mathfrak{m} \left| S \right| \).

In order to understand the physical meaning of the imaginary component of power, \( Q \), we shall study three examples, the first is an inductance \( L \) circuit, the second a capacitor \( C \), and the third an RLC circuit. First the inductive circuit with inductance \( L \). The impedance is \( Z = j \omega L \). Using rms values we have:

\[ S = VI^* = Z \left| I \right|^2 = j \omega L \left| I \right|^2 . \]

Thus, it is clear that \( P = 0 \), while \( Q = \mathfrak{m} \left| S \right| = \omega L \left| I \right|^2 \). Next we compute the instantaneous power and compare. Let us assume the current is given by \( i(t) = \sqrt{2} \left| I \right| \cos \omega t + \theta \), then the voltage on the inductor would be \( v(t) = L \frac{dI}{dt} = \sqrt{2} \omega L \left| I \right| \sin \omega t + \theta \). Thus the instantaneous power is given by:

\[ p(t) = v(t)i(t) = -2 \omega L \left| I \right|^2 \sin(\omega t + \theta) \cos(\omega t + \theta) = -\omega L \left| I \right|^2 \sin 2(\omega t + \theta) . \]

And so, \(-Q\) is the amplitude of the sinusoidal power (whose average value is zero) at frequency \( 2\omega \) in the inductor.

Next we take the example of the capacitor \( C \). Now \( Z = -j / \omega C \). Similar to the analysis above, we have:

\[ S = VI^* = V \left| V \right| Z^* = \left| V \right|^2 / Z^* = -j \left| V \right|^2 \omega C . \]

Again, we have \( P = 0 \), while \( Q = \mathfrak{m} \left| S \right| = -\left| V \right|^2 \omega C \). As was done earlier, and without loss of generality, let us assume the voltage (not the current) is given by:

\[ v(t) = \sqrt{2} \left| V \right| \cos \omega t + \theta \].

Hence the current is \( i(t) = C \frac{dv}{dt} = -\sqrt{2} \omega C \left| V \right| \sin \omega t + \theta \). The instantaneous power is given by:

\[ p(t) = v(t)i(t) = -2 \omega C \left| V \right|^2 \sin(\omega t + \theta) \cos(\omega t + \theta) = -\omega C \left| V \right|^2 \sin 2(\omega t + \theta) . \]

Once again we note that \( p(t) = Q \sin 2 \omega t + \theta \). In this case, \( Q \) is the amplitude of the sinusoidal power (whose average value is zero) at frequency \( 2\omega \) in the capacitor.

As a final example, we shall take an RLC circuit. Now \( Z = R + jX = \left| Z \right| \angle Z \) where \( X \) is either negative or positive depending on whether the circuit is capacitive or inductive (this depends on the values of \( R \)'s and \( C \)'s and \( \omega \)). Now the instantaneous power is given by:

\[ S = VI^* = Z \left| I \right|^2 = R \left| I \right|^2 + jX \left| I \right|^2 = P + jQ \].

Thus:

\[ P = \mathfrak{R} \left| S \right| = \left| Z \right| \left| I \right|^2 \cos (\angle Z) \]

and \( Q = X \left| I \right|^2 = \left| Z \right| \left| I \right|^2 \sin (\angle Z) \). Next we compute the instantaneous power. Without
loss of generality, let \( i(t) = \sqrt{2} |i| \cos \omega t \), then \( v(t) = \sqrt{2} |Z| |I|^2 \cos (\omega t + \angle Z) \). The power is:

\[
p(t) = v(t) i(t) = 2 |Z| |I|^2 \cos (\omega t) \cos (\omega t + \angle Z) .
\]

Using the trigonometric identity

\[
\cos x \cos y = \frac{1}{2} [\cos (x + y) + \cos (x - y)]
\]

we have

\[
p(t) = |Z| |I|^2 [\cos (\angle Z) + \cos (2\omega t + \angle Z)]
\]

or,

\[
p(t) = |Z| |I|^2 [\cos (\angle Z) + \cos (2\omega t) \cos (\angle Z) - \sin (2\omega t) \sin (\angle Z)]
\]

which can be simplified to:

\[
p(t) = P (1 + \cos 2\omega t) - Q \sin 2\omega t .
\]

Now consider the same RLC with all three elements present in series (i.e. we shall not reduce it into its equivalent RL or RC as was done above). In this case

\[
Z = R + j\omega L + 1/ j\omega C .
\]

We also know that

\[
P = R |I|^2 , \quad Q_L = \omega L |I|^2 \quad \text{and} \quad Q_C = -|I|^2 / \omega C .
\]

Thus

\[
Q = Q_L + Q_C ,
\]

and the expression at end of last paragraph can be written as:

\[
p(t) = P (1 + \cos 2\omega t) - Q_L + Q_C \sin 2\omega t .
\]

This is very interesting since it is now possible that \( Q \) can be zero. This is true when \( Q_L = -Q_C \) or when \( X_L = X_C \). This is a condition when we have resonance and the capacitance cancels the inductance at the frequency \( \omega \). This is the case when \( LC = 1/\omega^2 \).

The following theorem is stated without proof.

**Conservation of Complex Power.** In a power system supplied by sources at the same frequency, the complex power supplied by the sources is exactly equal to the complex power absorbed by all the other components of the system. Another way of stating this is that the sum of complex powers for all components of a power system (sources and loads) is equal to zero.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Termination</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td>Complex power</td>
<td>VA, kVA, MVA</td>
</tr>
<tr>
<td></td>
<td>S</td>
<td></td>
</tr>
<tr>
<td>P</td>
<td>Average or real or active power</td>
<td>W, kW, MW</td>
</tr>
<tr>
<td>Q</td>
<td>Reactive power</td>
<td>var, kvar, Mvar</td>
</tr>
</tbody>
</table>

The following theorem is stated without proof.

**Conservation of Complex Power.** In a power system supplied by sources at the same frequency, the complex power supplied by the sources is exactly equal to the complex power absorbed by all the other components of the system. Another way of stating this is that the sum of complex powers for all components of a power system (sources and loads) is equal to zero.
It is noted that a capacitor, for example, is a generator of Q, i.e. a capacitor does not consume Var’s, it generates them (its Q is negative). An inductor does consume Var’s, its Q is positive. A resistor always consumes Watts. A synchronous machine can consume or generate P and/or Q. Thus there are four different possibilities for a synchronous machine.

Now we study a very important example on the flow of power on a short transmission line. The example can be generalized to power flow between any two buses with an impedance connecting them. Let us assume that the line has an impedance of $Z = R + jX = |Z|e^{j\theta}$ Ohm. At the left end let us assume a sinusoidal voltage of value $E_1 = V_1 e^{j\delta_1}$ while on the right end the voltage is $E_2 = V_2 e^{j\delta_2}$. We shall solve for the power flow across the line. The complex power flowing from left to right at the left end is denoted by $S_{12}$. The complex power flowing from right to left at the right end is denoted by $S_{21}$. [Note that the complex power from right to left at the left node is $-S_{12}$ and the complex power from left to right at the right node is $-S_{21}$] Solving for the current $I$, we have: $I = (E_1 - E_2)/Z$. It is noted that all the quantities in these equations are, in general, complex. The transmission line is shown in the figure below. It is easy to see that the complex power flowing from left to right is given by:

$$S_{12} = E_1 I^* = E_1 [(E_1 - E_2)/Z]^*, \text{ where } I^* \text{ indicates the complex conjugate of } I. \text{ This can be further simplified as:}$$

$$S_{12} = \frac{V_1^2}{Z} - \frac{E_1 E_2^*}{Z} = \frac{V_1^2}{|Z|} e^{-j\theta} - \frac{V_1 V_2}{|Z|} e^{-j(\theta + \delta_1 - \delta_2)}. \text{ If we let } \delta = \delta_1 - \delta_2, \text{ then we have:}$$

$$S_{12} = \frac{V_1^2}{|Z|} e^{-j\theta} - \frac{V_1 V_2}{|Z|} e^{-j(\theta + \delta)}. \text{ In a similar manner, the power going from right to left is given by the very similar expression:}$$

$$S_{21} = \frac{V_2^2}{|Z|} e^{-j\theta} - \frac{V_1 V_2}{|Z|} e^{-j(\theta - \delta)}. \text{ Note that } S_{12} \text{ is the complex power flowing from left to right at the left node. } S_{21} \text{ is the complex power from right to left at the right node. The complex power received at the right node is } -S_{21}. \text{ Note that the complex power received at the right end is not equal to the complex power sent at the left end. The difference is due to the complex power received by the line impedance } Z.$$
\[ Q_{21} = \frac{V_2^2}{X} - \frac{V_1 V_1}{X} \cos \delta. \]

It is noted that \( P_{12} = -P_{21} \) while \( Q_{12} \neq -Q_{21} \). The difference between \( Q \) sent from the left and \( Q \) received on the right \((Q_{12} + Q_{21})\) is the reactive power used by the line. Also, the maximum real power the line can transfer (at \( \delta = \pm 90^\circ \)) is \( V_1 V_2 / X \), which is much higher than the power real lines are subjected to (i.e. |\( \delta \)|\( << 90^\circ \), and usually |\( \delta \)|\( \leq 10^\circ \)).

As another example, consider the RC circuit shown where the input is a sinusoid. The reactance of the capacitor is \( X_C = 1 / (\omega C) \). The impedance of the capacitor would be \( Z_C = 1 / (j\omega C) = -j / (\omega C) \). The voltage on the capacitor would be given by:

\[ E_C = \frac{Z_C}{R + Z_C} E_1. \]

The voltage transfer ratio would be:

\[ \frac{E_C}{E_1} = \frac{Z_C}{1 + j\omega RC}. \]

This is to be compared with the Laplace transform of the voltage transfer ratio, namely:

\[ \frac{1}{1 + s\tau} \]

where \( \tau \) is the time constant \( \tau = RC \). The current is given by:

\[ I = \frac{E_1}{R - j/(\omega C)} = \frac{E_1}{R + Z_C}. \]

We know the power in the resistor would be: \( |I|^2 R \). Let us find this power by using the equation for complex power \( S = VI^* \) applied to the resistor: \( S_R = (E_1 - E_C)I^* = E_1 - \frac{Z_C}{R + Z_C} E_1 I^* \). This is expanded and simplified thus:

\[ S_R = E_1 \left( 1 - \frac{Z_C}{R + Z_C} \right) I^* = E_1 \left( 1 - \frac{Z_C}{R + Z_C} \right) \frac{E_1}{R + Z_C} I^* \].

This is further reduced to:

\[ S_R = |E_1|^2 \frac{R}{R + Z_C} \left( 1 - \frac{1}{R + Z_C} \right) = |E_1|^2 R = P_R \].

This is a very interesting demonstration that \( S = VI^* \) applies to a resistor! Next we compute the complex power on the capacitor (which should be a pure imaginary number): \( S_C = E_C I_C^* \).

This is expanded thus:

\[ S_C = E_C I_C^* = \frac{Z_C}{R + Z_C} E_1 \frac{E_1}{R + Z_C} I^* = Z_C |I|^2 = -\frac{j}{\omega C} |I|^2 X. \]

We see that the capacitor “generates” vars and does not “consume” them (as seen from the minus sign). The total complex power consumed by this circuit is then given by:

\[ S = S_R + S_C = |I|^2 R - jX. \]

As an exercise, please prove this last expression in a different way by expanding \( S = E_1 I^* \).
An inductor would have produced a similar result for complex power except it would have been a positive imaginary number, thus an inductor “consumes” vars. The expression for an inductor would have been: \( S_L = j|I_i|^2 X_L = j\omega L|I_i|^2 \).

**The power triangle.** Let a circuit of impedance \( Z \angle \theta \) be connected to a supply whose voltage is \( V \angle 0 \). The current would then be given by the ratio of voltage and impedance, \( I = V \angle 0 / Z \angle \theta = \left| V \right| / \left| Z \right| \angle - \theta = \left| I \right| \angle - \theta \). The complex power would be \( S = VI^* = \left| V \right| \left| I \right| \angle \theta = \left| S \right| \angle \theta = P + jQ \). The real power is determined from the equation \( P = \left| S \right| \cos \theta = \left| V \right| \left| I \right| \cos \theta \), and the imaginary power is determined from the equation \( Q = \left| S \right| \sin \theta = \left| V \right| \left| I \right| \sin \theta \). The three vector quantities \( P, jQ \) and \( S \) form a triangle as shown in the figure on the right. The angle \( \theta \) is the angle of the impedance of the network, namely, \( \theta = \angle Z \).

**Delta-Y load transformation.** Assuming a symmetrical load configuration with delta loads equal to \( Z_\Delta \), then the equivalent Y loads would be \( Z_Y = Z_\Delta / 3 \). This is useful in transforming Y-loads to Delta-loads and vice-versa. Keep in mind that this equivalence is only at the terminals. If variables are needed inside the load, then one has to go back to the original circuit before the transformation. It would be a good exercise for the student to prove this transformation equation.

**Delta-Y source transformation.** Assuming a balanced \( 3\phi \) positive sequence source, the Delta to Y transformation (and vice-versa) are as follows, where by definition \( V_{ab} \equiv V_a - V_b \) and \( V_a \equiv V_{an} \):

\[
V_{an} = V_a = \frac{1}{\sqrt{3}} e^{-j\pi/6} V_{ab} \quad \text{and} \quad V_{ab} = \sqrt{3} e^{j\pi/6} V_{an} = \sqrt{3} e^{j\pi/6} V_a .
\]

Similar expressions are valid for the other phases. If these were negative sequence instead of positive sequence, the applicable diagram is shown below and the relationships become:

\[
V_{an} = V_a = \frac{1}{\sqrt{3}} e^{j\pi/6} V_{ab} \quad \text{and} \quad V_{ab} = \sqrt{3} e^{-j\pi/6} V_{an} = \sqrt{3} e^{-j\pi/6} V_a .
\]

**Per phase analysis.** Given a three phase electrical system with various loads and sources and assuming all the loads, connections and sources are balanced three phase, and there are no mutual couplings between the phases, then it is possible to analyze the system by studying only a single phase “equivalent”. The main principle which allows per phase analysis follows. Under the assumptions
just made, the neutrals are all at the same potential, hence they may be connected to
together, thus separating the circuit into three completely decoupled circuits, one for each
phase. The three circuits would be identical, except for the phase differential of 120° or
240° for the other two phases (we usually assume that phase ‘a’ is being solved). If there
is a Delta it can be changed to a Y, thus a neutral is available everywhere.

Outline of per phase analysis. We assume that loads and sources are balanced three
phase. We also assume that there are no mutuals between phases. Under these assump-
tions we proceed to solve the as follows:

1. Convert all deltas (sources and loads) to equivalent Y.
2. Take phase ‘a’ only, and solve its variables.
3. The phase ‘b’ and ‘c’ variables are found after the proper phase shift.
4. If needed, re-transform to the deltas converted in step 1.

Balanced three phase power. It is clear from the above that if “per phase” analysis can
be performed, then the results for each phase is similar, thus the power per phase is the
same, and the total power is three times that per phase. Thus if all voltages are referred to
the neutral, then: \( S_{3p} = V_a I_a^* + V_b I_b^* + V_c I_c^* \). Expressing the ‘b’ and ‘c’ phases in terms of
the ‘a’ phase we have: \( S_{3p} = V_a I_a^* + V_a e^{-j2\pi/3} I_a^* e^{j2\pi/3} + V_a e^{j2\pi/3} I_a^* e^{-j2\pi/3} = 3V_a I_a^* = 3S \) where
\( S \) is the complex power per phase.

From page 3 we have  
\[ p(t) = \frac{1}{2} V_m I_m [\cos \theta_v - \theta_i + \cos 2\omega t + \theta_v + \theta_i] \] for one phase.

Figuring the three phase instantaneous power we have:
\[ p_{3p}(t) = v_a(t) i_a(t) + v_b(t) i_b(t) + v_c(t) i_c(t) \]
\[ = |V| I \left[ \cos \phi + \cos 2\omega t + \theta_v + \theta_i \right] \]
\[ + |V| I \left[ \cos \phi + \cos 2\omega t + \theta_v + \theta_i - 4\pi / 3 \right] \]
\[ + |V| I \left[ \cos \phi + \cos 2\omega t + \theta_v + \theta_i + 4\pi / 3 \right] \]
\[ = 3|V| I |\cos \phi| \]
\[ = 3P \]

It is easy to see that the double frequency terms (in the square brackets above) add to zero
by examining their sum as phasors. This interesting result shows that the three phase
power is not a function of time! I.e. it is a constant. This is one of the main advantages
of using three phase power over single phase power. Please take a close look at the equa-
tion above, the LHS is a function of time while the RHS is not!

The power of each phase above has a double frequency component as well as a ‘steady’
component. When all three phases are added up, the double frequency components van-
ish leaving a constant steady component.
Next we study the result of balanced three phase currents in balanced three phase windings in the stator of a synchronous machine. For simplicity we show each of the three windings as a single turn whose cross section is shown in the figure below. For example, the winding a-a' would have a flux along the x-axis. If the current $I_a$ is positive (into the paper at the ‘x’ and out of the paper at the dot), then the flux would be along the positive x-axis. If we assume that the flux has a sinusoidal space distribution around the perimeter of the machine, then the flux equation as a function of $\alpha$ would be given by: $F_a = f_a(t) \cos \alpha$, where $f_a(t)$ is the value of the flux for phase ‘a’ at time $t$. Similarly for the other two phases we have: $F_b = f_b(t) \cos \alpha - 2\pi/3$ and $F_c = f_c(t) \cos \alpha + 2\pi/3$. These expressions give the flux value at the position $\alpha$ shown below.

Now, the value of the flux is directly proportional to the current in phase ‘a’, or: $f_a(t) = K I_a \cos(\omega t + \phi)$ where $\phi$ is an arbitrary angle for the current. Similarly, for the fluxes of the other phases we have: $f_b(t) = K I_a \cos(\omega t + \phi - 2\pi/3)$ and $f_c(t) = K I_a \cos(\omega t + \phi + 2\pi/3)$. These expressions represent a pulsating or standing wave along the flux axis of each coil. Thus we now have the following for the three fluxes at angle $\alpha$:

- $F_a = F_m \cos \alpha \cos \omega t + \phi$ where $F_m = K I_a$,
- $F_b = F_m \cos \alpha - 2\pi/3 \cos \omega t + \phi - 2\pi/3$,
- $F_c = F_m \cos \alpha + 2\pi/3 \cos \omega t + \phi + 2\pi/3$. Thus the total flux at angle $\alpha$ would be: $F_{ar} = F_a + F_b + F_c$ where “ar” stands for armature reaction, i.e. the flux due to currents in the armature also known in this case as the stator. This sum can be expressed as:

$$F_{ar} = \frac{F_m}{2} \cos \alpha + \omega t + \phi + \cos \alpha - \omega t - \phi$$

$$+ \frac{F_m}{2} \cos \alpha + \omega t + \phi - 4\pi/3 + \cos \alpha - \omega t - \phi$$

$$+ \frac{F_m}{2} \cos \alpha + \omega t + \phi + 4\pi/3 + \cos \alpha - \omega t - \phi$$

Note that the sum of the three first terms in each of the bracketed expressions above is zero. Thus we have: $F_{ar} = \frac{3F_m}{2} \cos \alpha - \omega t - \phi$. This expression has very far reaching consequences. It is a function of time as well as space (peripheral angle $\alpha$). It is called a travelling wave. Thus the sum of the three pulsating waves became a travelling wave. This wave travels around the perimeter at speed $\omega t$, the same speed and direction as a rotor which would have generated these currents had this been a generator. This travelling wave is a travelling sinusoid in space rotating around at the synchronous speed $\omega$. 


This demonstrates that generator action and motor action are reversible, i.e. a generator can be a motor and vice-vera.