1.

With scalar $a > 0$, consider the signal $g_a : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$g_a(t) := \cos (2\pi t) + \sin (2\pi at), \quad t \in \mathbb{R}.$$

1.a. For each $T > 0$ we have

$$\int_{-T}^{T} |g_a(t)|^2 dt = \int_{-T}^{T} |\cos (2\pi t) + \sin (2\pi at)|^2 dt = \int_{-T}^{T} (|\cos (2\pi t)|^2 + 2 \cos (2\pi t) \sin (2\pi at) + |\sin (2\pi at)|^2) dt. \quad (1.1)$$

With the help of standard trigonometric identities, elementary calculations yield

$$\int_{-T}^{T} |\cos (2\pi t)|^2 dt = \frac{1}{2} \int_{-T}^{T} (1 + \cos (4\pi t)) dt = T + \frac{\sin (4\pi T)}{4\pi} \quad (1.2)$$

and

$$\int_{-T}^{T} |\sin (2\pi at)|^2 dt = \frac{1}{2} \int_{-T}^{T} (1 - \cos (4\pi at)) dt = T - \frac{\sin (4\pi aT)}{4\pi a}, \quad (1.3)$$

while

$$\int_{-T}^{T} \cos (2\pi t) \sin (2\pi at) dt = 0.$$
since the integrand has odd symmetry with respect to the origin. As a result we conclude that

\[ P_{g_a} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |g_a(t)|^2 dt = 1. \]

1.b. It is assumed that the signal \( g_a : \mathbb{R} \to \mathbb{R} \) gives rise to a Fourier series expansion of the form

\[ \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n a t}, \quad t \in \mathbb{R} \quad (1.4) \]

with Fourier coefficients \( \{c_n, n = 0, \pm 1, \ldots\} \). The signal \( g_a \) being defined on \( \mathbb{R} \), the existence of its Fourier series \( (??) \) implies that \( g_a \) must be periodic with period

\[ T := \frac{1}{a}. \]

But the signal \( t \to \sin (2\pi at) \) being itself periodic with period \( T \), it follows that the signal \( t \to \cos (2\pi t) \) must also be periodic with period \( T \). However, the signal \( t \to \cos (2\pi t) \) is itself periodic with period 1. These two requirements imply that \( T = \ell \) for some integer \( \ell = 1, 2, \ldots \), or equivalently

\[ a = \frac{1}{\ell}. \]

1.c. Under the condition \( a = \frac{1}{\ell} \) for some \( \ell = 1, \ldots \), we get the following: If \( \ell \neq 1 \), then

\[ g_a(t) = \frac{e^{2\pi it} + e^{-2\pi it}}{2} + \frac{e^{2\pi iat} - e^{-2\pi iat}}{2j} \]

\[ = \frac{e^{2\pi it} + e^{-2\pi it}}{2} + \frac{e^{2\pi iat} - e^{-2\pi iat}}{2j}, \quad t \in \mathbb{R} \]

so that

\[ c_1 = \frac{1}{2j}, \quad c_{-1} = -\frac{1}{2j} \quad \text{and} \quad c_\ell = c_{-\ell} = \frac{1}{2} \]

with all other Fourier coefficients being zero. If \( \ell = 1 \), then \( a = 1 \) and we get

\[ c_{\pm 1} = \frac{1}{2} \left( 1 \pm \frac{1}{j} \right) \]

with all other Fourier coefficients being zero. In either case we now conclude that

\[ P_{g_a} = \frac{1}{T} \int_{0}^{\frac{T}{2}} |g_a(t)|^2 dt \quad \text{[By periodicity]} \]

\[ = \sum_{n=-\infty}^{\infty} |c_n|^2 \quad \text{[By Parseval's Theorem for Fourier series]} \]

\[ = \begin{cases} \frac{1}{4} \left| 1 - \frac{1}{j} \right|^2 + \frac{1}{4} \left| 1 + \frac{1}{j} \right|^2 & \text{if } \ell = 1 \\ 2 \left| \frac{1}{2j} \right|^2 + 2 \left| \frac{1}{2} \right|^2 & \text{if } \ell \neq 1 \end{cases} \]

\[ = 1, \quad (1.5) \]

in agreement with the evaluation carried out in Part 1.a.
2. For each $a > 0$, consider the signal $h_a : \mathbb{R} \to \mathbb{R}$ given by

$$h_a(t) = e^{-a|t|}, \quad t \in \mathbb{R}.$$  

2.a. By now you should know that

$$H_a(f) = \frac{2a}{a^2 + (2\pi f)^2}, \quad f \in \mathbb{R}$$

and I look forward to seeing your calculations.

2.b. Note that $\lim_{a \downarrow 0} h_a(t) = 1$ for each $t$ in $\mathbb{R}$ and it is easily verified that

$$\lim_{a \downarrow 0} H_a(f) = \begin{cases} 0 & \text{if } f \neq 0 \\ \infty & \text{if } f = 0 \end{cases}$$

so that $\lim_{a \downarrow 0} H_a(f)$ can be viewed as a proxy for $\delta(f)$. Thus, we can construct a direct approximation argument on the way to establish the Fourier pairing $1 \iff \delta(f)$, namely

$$h_a(t) \downarrow (a \downarrow 0) \iff H_a(f) \downarrow (a \downarrow 0) \iff \delta(f)$$
3. Consider the function \( v : [-\frac{1}{2}, \frac{1}{2}] \rightarrow \mathbb{R} \) given by
\[
v(t) = |t|, \quad |t| \leq \frac{1}{2}.
\]

3.a. Note that
\[
v_k = \int_{-\frac{1}{2}}^{\frac{1}{2}} |t| e^{-j2\pi k t} \, dt, \quad k = 0, \pm 1, \pm 2, \ldots
\]
so that
\[
v_{-k} = v_k, \quad k = 1, 2, \ldots
\]
with
\[
v_0 = \int_{-\frac{1}{2}}^{\frac{1}{2}} |t| \, dt = 2 \int_{0}^{\frac{1}{2}} t \, dt = \frac{1}{4}.
\]

Now for each \( k = 1, 2, \ldots \), we have
\[
v_k = \int_{-\frac{1}{2}}^{\frac{1}{2}} |t| e^{-j2\pi k t} \, dt = a_k + b_k
\]
where
\[
a_k := \int_{0}^{\frac{1}{2}} |t| e^{-j2\pi k t} \, dt = \int_{0}^{\frac{1}{2}} t e^{-j2\pi k t} \, dt
\]
and
\[
b_k := \int_{-\frac{1}{2}}^{0} |t| e^{-j2\pi k t} \, dt = -\int_{-\frac{1}{2}}^{0} t e^{-j2\pi k t} \, dt.
\]

It is clear that
\[
b_k = -\int_{-\frac{1}{2}}^{0} t e^{-j2\pi k t} \, dt
\]
\[
= -\int_{\frac{1}{2}}^{0} s e^{j2\pi k s} \, ds \quad \text{[Change of variable } t = -s]\]
\[
= \int_{0}^{\frac{1}{2}} s e^{j2\pi k s} \, ds = a_k^*.
\]
Integration by parts gives
\[
    a_k = \int_0^{1/2} te^{-j2\pi kt} \, dt \\
    = \int_0^{1/2} t \left( \frac{e^{-j2\pi kt}}{-j2\pi k} \right)' \, dt \\
    = \left[ t \cdot \left( \frac{e^{-j2\pi kt}}{-j2\pi k} \right) \right]_0^{1/2} - \int_0^{1/2} \frac{e^{-j2\pi kt}}{-j2\pi k} \, dt \\
    = \frac{1}{2} \left( e^{-j\pi k} \right) - \frac{e^{-j2\pi kt}}{-j2\pi k} \left|_0^{1/2} \right. \\
    = \frac{1}{2} \left( e^{-j\pi k} \right) - \frac{e^{-j\pi k} - 1}{(-j2\pi k)^2} \\
    = \frac{1}{2} \left( (-1)^k \right) - \frac{(-1)^k - 1}{(-j2\pi k)^2}.
\]

It is now immediate that
\[
b_k = a_k^* = \frac{1}{2} \left( \frac{(-1)^k}{j2\pi k} \right) - \frac{(-1)^k - 1}{(-j2\pi k)^2}
\]
so that
\[
v_k = a_k + b_k = -2 \frac{(-1)^k - 1}{(-j2\pi k)^2} = \frac{(-1)^k - 1}{2(\pi k)^2}.
\]
Finally, for each \( k = 1, 2, \ldots \), we have
\[
v_k = \begin{cases} 
    0 & \text{if } k \text{ even} \\
    -\frac{1}{(\pi k)^2} & \text{if } k \text{ odd}.
\end{cases}
\]
Hence,
\[
v(t) = \sum_{k=-\infty}^{\infty} v_k e^{j2\pi kt} \\
    = v_0 + \sum_{k=1}^{\infty} v_k \left( e^{j2\pi kt} + e^{-j2\pi kt} \right) \\
    = \frac{1}{4} + 2 \sum_{k=1}^{\infty} v_k \cos (2\pi kt) \\
    = \frac{1}{4} + 2 \sum_{\ell=0}^{\infty} v_{2\ell+1} \cos (2\pi (2\ell + 1)t) \\
    = \frac{1}{4} - 2 \sum_{\ell=0}^{\infty} \frac{1}{\pi(2\ell + 1)^2} \cos (2\pi (2\ell + 1)t) \\
    = \frac{1}{4} - \frac{2}{\pi^2} \sum_{\ell=0}^{\infty} \frac{1}{(2\ell + 1)^2} \cos (2\pi (2\ell + 1)t), \quad t \in \mathbb{R}.
\] (1.6)
3.b. By Parseval’s Theorem for Fourier series we know that
\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} |t|^2 dt = \sum_{k=-\infty}^{\infty} |v_k|^2.
\]
Noting that
\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} |t|^2 dt = \frac{1}{3} \int_{-\frac{1}{2}}^{\frac{1}{2}} 3t^2 dt = \frac{1}{3} \left( 2 \left( \frac{1}{2} \right)^3 \right) = \frac{1}{12},
\]
we conclude that \( I(v) = \frac{1}{2} \).

3.c. The calculations are straightforward: Note that
\[
\sum_{k=-\infty}^{\infty} |v_k|^2 = |v_0|^2 + \sum_{k=-\infty}^{\infty} |v_k|^2
\]
\[
= \frac{1}{16} + 2 \sum_{k=1}^{\infty} |v_k|^2
\]
\[
= \frac{1}{16} + 2 \sum_{\ell=0}^{\infty} |v_{2\ell+1}|^2
\]
\[
= \frac{1}{16} + 2 \sum_{\ell=0}^{\infty} \frac{1}{\pi (2\ell + 1)^4} = \frac{1}{16} + \frac{2}{\pi^4} \sum_{\ell=0}^{\infty} \frac{1}{(2\ell + 1)^4} \tag{1.7}
\]
and Part 3.b yields
\[
\frac{1}{12} = \frac{1}{16} + \frac{2}{\pi^4} \sum_{\ell=0}^{\infty} \frac{1}{(2\ell + 1)^4}.
\]
Solving for \( \pi^4 \) we get
\[
\pi^4 = 96 \cdot \sum_{\ell=0}^{\infty} \frac{1}{(2\ell + 1)^4}.
\]
4. It is high time to compute various integrals using the properties of the Fourier transform: With the function \( h : \mathbb{R} \to \mathbb{R} \) being defined by

\[
h(t) = \begin{cases} 
1 & \text{if } |t| \leq 1 \\
0 & \text{if } |t| > 1,
\end{cases}
\]

we have

\[ h(t) \iff H(f) \]

with

\[ H(f) = 2 \cdot \frac{\sin(2\pi f)}{2\pi f}, \quad f \in \mathbb{R}, \]

and by duality we conclude that

\[ H(-t) \iff h(f). \]

4.a. With this in mind, note that

\[
2I = \int_{\mathbb{R}} \left( \frac{\sin t}{t} \right)^2 \, dt \\
= 2\pi \int_{\mathbb{R}} \left( \frac{\sin(2\pi s)}{2\pi s} \right)^2 \, ds \quad \text{[Change of variable } t = 2\pi s] \\
= \frac{\pi}{2} \int_{\mathbb{R}} \left( 2 \cdot \frac{\sin(2\pi f)}{2\pi f} \right)^2 \, df \quad \text{[Change of variable } s = f] \\
= \frac{\pi}{2} \int_{\mathbb{R}} |H(f)|^2 \, df \\
= \frac{\pi}{2} \int_{\mathbb{R}} |h(t)|^2 \, dt \quad \text{[Parseval’s Theorem for Fourier transforms]} \\
= \frac{\pi}{2} \int_{-1}^{1} \, dt \\
= \pi, \quad \text{hence } I = \frac{\pi}{2}. \tag{1.8}
\]
4.b. Note that

\[
I(a) = \int_{\mathbb{R}} e^{-a|t|} \frac{\sin t}{t} dt
\]

\[
= 2\pi \int_{\mathbb{R}} e^{-2\pi a|s|} \frac{\sin (2\pi s)}{2\pi s} ds \quad \text{[Change of variable } t = 2\pi s]\]

\[
= \pi \int_{\mathbb{R}} e^{-b|s|} \left( \frac{2 \sin (2\pi s)}{2\pi s} \right) ds \quad \text{[Set } b = 2\pi a]\]

\[
= \pi \int_{\mathbb{R}} e^{-b|s|} \cdot H(-s) ds
\]

\[
= \pi \int_{\mathbb{R}} \frac{2b}{b^2 + (2\pi f)^2} \cdot h(f) df \quad \text{[By Parseval’s Theorem for Fourier transforms]}\]

\[
= \pi \int_{-1}^{1} \frac{2b}{b^2 + (2\pi f)^2} df
\]

\[
= \frac{2\pi}{b} \int_{-1}^{1} \frac{1}{1 + \left( \frac{2\pi}{b} f \right)^2} df
\]

\[
= \frac{2\pi a}{b} \int_{-a^{-1}}^{a^{-1}} \frac{1}{1 + x^2} dx \quad \text{[Change of variable } x = \frac{2\pi}{b} f = \frac{f}{a}]\]

\[
= 2\text{Arctan} \left( a^{-1} \right). \tag{1.9}
\]

4.c. Note that

\[
J(a) = \int_{0}^{\infty} e^{-at} \frac{\sin t}{t} dt
\]

\[
= \int_{0}^{\infty} e^{-a|t|} \frac{\sin t}{t} dt
\]

\[
= \frac{1}{2} I(a) \tag{1.10}
\]

since

\[
\int_{-\infty}^{0} e^{-a|t|} \frac{\sin t}{t} dt = \int_{0}^{\infty} e^{-a|t|} \frac{\sin t}{t} dt
\]

by symmetry, and the conclusion

\[
J(a) = \text{Arctan} \left( a^{-1} \right)
\]

follows.

As stated in the hint to this question, there are many different ways to compute these integrals. In particular you should also be aware of the fact that if \( g_1, g_2 : \mathbb{R} \to \mathbb{C} \) are finite energy signals with Fourier transforms \( G_1, G_2 : \mathbb{R} \to \mathbb{C} \), then

\[
\int_{\mathbb{R}} g_1(t)g_2(t) dt = \langle g_1 \ast g_2 \rangle (0) \tag{1.11}
\]
The signal $g_1 \ast g_2$ has a Fourier transform given by $G_1 \cdot G_2$, and the inverse Fourier transform yields

$$ (g_1 \ast g_2)(t) = \int_{\mathbb{R}} G_1(f) \cdot G_2(f) e^{j2\pi ft} df $$

so that

$$ \int_{\mathbb{R}} g_1(t)g_2(t)dt = (g_1 \ast g_2)(0) = \int_{\mathbb{R}} G_1(f) \cdot G_2(f) df. $$