COMMUNICATION SYSTEMS

HOMEWORK # 2 – An Answer to Ex. 10

Consider a periodic signal $g : \mathbb{R} \rightarrow \mathbb{C}$ with period $T > 0$, i.e.,

$$g(t + T) = g(t), \quad t \in \mathbb{R}.$$ 

Assume that

$$\int_0^T |g(t)| dt < \infty.$$ 

Show that the Fourier coefficients of $g$ can be computed as

$$c_n = \frac{1}{T} \int_a^{a+T} g(t)e^{-j\frac{2\pi}{T} t} dt, \quad n = 0, \pm 1, \pm 2, \ldots$$ 

regardless of the value of $a$ in $\mathbb{R}$.

For any scalar $a$ in $\mathbb{R}$, it is always possible to write $a = \ell T + \alpha$ for some $\ell = 0, \pm 1, \ldots$ and $0 \leq \alpha < T$ – Note that $\ell$ and $\alpha$ are uniquely determined by $a$ once $T > 0$ is given. Now, for any mapping $h : \mathbb{R} \rightarrow \mathbb{C}$ which is periodic with period $T$, whenever the integrability condition

$$\int_0^T |h(t)| dt < \infty$$

holds, we have

$$\int_a^{a+T} h(t) dt = \int_0^T h(t) dt.$$  \hspace{1cm} (1.1)
Indeed, we have
\[
\int_a^{a+T} h(t) \, dt = \int_0^T h(a+s) \, ds \quad \text{[Change of variable: } t = a+s]\]
\[
= \int_0^T h(\ell T + \alpha + s) \, ds \quad \text{[Use the fact that } a = \ell T + \alpha]\]
\[
= \int_0^T h(\alpha + s) \, ds \quad \text{[Use the fact that } h \text{ is periodic of period } T]\]
\[
= \int_0^a h(\alpha + s) \, ds + \int_\alpha^T h(\alpha + s) \, ds \]
\[
= \int_0^a h(\alpha + s) \, ds + \int_0^{T-\alpha} h(x) \, dx \quad \text{[Change of variable } x = \alpha + s]\]
\[
= \int_{T-\alpha}^T h(y) \, dy + \int_0^{T-\alpha} h(x) \, dx \quad \text{[Change of variable } y = T - \alpha + s]\]
\[
= \int_0^T h(x) \, dx \quad (1.2)
\]
and (1.1) is established.

The exercise is solved by applying (1.1) to the functions
\[
h_n(t) = g(t)e^{-j2\pi \frac{n}{T} t}, \quad t \in \mathbb{R}, \quad n = 0, \pm 1, \pm 2, \ldots
\]
where \( g : \mathbb{R} \to \mathbb{C} \) is periodic with period \( T \) and satisfies the integrability condition
\[
\int_0^T |g(t)| \, dt < \infty.
\]
Indeed, for each \( n = 0, \pm 1, \pm 2, \ldots \), the function \( h_n : \mathbb{R} \to \mathbb{C} \) is periodic of period \( T \) since
\[
h_n(t + T) = g(t + T)e^{-j2\pi \frac{n}{T} (t+T)}
\]
\[
= g(t)e^{-j2\pi \frac{n}{T} (t+T)} \quad \text{[Recall that } g \text{ is periodic with period } T]\]
\[
= g(t)e^{-j2\pi n e^{-j2\pi \frac{n}{T} t}}
\]
\[
= g(t)e^{j2\pi \frac{n}{T} t}
\]
\[
= h_n(t), \quad n = 0, \pm 1, \pm 2, \ldots \quad (1.3)
\]
Also
\[
\int_0^T |h_n(t)| \, dt = \int_0^T |g(t)| \, dt < \infty
\]
and the integrability condition is satisfied.