#1. Note that

\[
\cos 2\theta = \cos^2 \theta - \sin^2 \theta, \quad \theta \in \mathbb{R}
\]

so that

\[
\cos^2 \theta = \cos 2\theta + \sin^2 \theta = \cos 2\theta + 1 - \cos^2 \theta
\]

\[
\cos^2 \theta = \frac{1 + \cos 2\theta}{2}, \quad \theta \in \mathbb{R}.
\]

Therefore,

\[
\cos^3 \theta = \cos^2 \theta \cdot \cos \theta
\]

\[
= \frac{\cos \theta + \cos \theta \cos 2\theta}{2}
\]

\[
= \frac{\cos \theta + (\cos (\theta + 2\theta) + \cos (\theta - 2\theta))/2}{2}
\]

\[
= \frac{1}{4} (\cos 3\theta + 3\cos \theta), \quad \theta \in \mathbb{R}.
\]

4. If \( y(t) = A_c \cdot m(t) \cdot \cos^3 (2\pi ft) \), then

\[
y(t) = \frac{A_c}{4} \cdot m(t) \left( \cos (3 \cdot 2\pi ft) + 3 \cos (2\pi ft) \right), \quad t \in \mathbb{R}
\]

so that

\[
y(\theta) = \frac{3A_c}{8} \left( M(f+\theta) + M(f-\theta) \right) + \frac{A_c}{8} \left( M(f+3\theta) + M(f-3\theta) \right), \quad \theta \in \mathbb{R}
\]
Pictorially, with $m \in LP(W)$ where $W < f_c$.

To be eliminated.

$\theta(t) = A_c \cos^3(2\pi f_c t)$

Where $A$ and $A_c$ are related by $A = \frac{3A_c}{4}$, and the BP filter $\in BP(f_c, W)$ (i.e., its passband is $|\pm f_c - f| \leq W$).

If $\theta(t) = \cos(2\pi f_c t)$, then

$y(t) = A_c m(t) \cos^2(2\pi f_c t)$

$= \frac{A_c}{2} m(t) (1 + \cos(4\pi f_c t)), \quad t \in \mathbb{R}$

So that

$y(t) = \frac{A_c}{4} \left( M(b + f_c) + M(b - f_c) \right) + \frac{A_c}{4} \left( M(b + 2f_c) + M(b - 2f_c) \right)$.
This signal \( y \) has no frequency content at \( f = \pm f_c \), so that no amount of linear filtering (applied to \( y \)) can result in a signal that would have frequency content at \( f = \pm f_c \) as would be the case for \( x(t) = A m(t) \cos (2\pi f t) \) with \( m \in \text{LP}(W) \).

#2. Note that

\[
m(t) = 10 \cos (1000\pi t) + 5 \cos (1500\pi t) \\
= 10 \cos (2\pi (500) t) + 5 \cos (2\pi (-750) t) \\
= 10 \cos (2\pi (f_c t)) + 5 \cos (2\pi (3f_c) t), \text{ EIR}
\]

with \( f_c = 250 \).

a/ The signal \( m \) is bandlimited, with \( m \in \text{LP}(W) \) and

\[
W = 3f_c = 750 \text{ (Hz)}
\]

so at minimum, we should take the sampling rate \( f_s = \frac{1}{T} \) to be at least the Nyquist rate of \( m \), namely

\[
f_{\text{Nyq}} = 2W = 6f_c = 1500 \text{ Hz}
\]

Of course, with DM, the requirement is \( f_s \gg f_{\text{Nyq}} = 1500 \), or equivalently,
\[ T_s \ll (1,500)^{-1} \text{ sec} \]

\( b_\text{) Slope overload avoided if} \)

\[
\frac{\Delta}{T_s} \geq \max \left\{ \left| \frac{dm(t)}{dt} \right|, \quad t \in I \right\}
\]

where \( I \) is the interval where the signal is observed, e.g., say \( I = IR \) for simplicity (As will be seen shortly, the specific interval plays no role).

But

\[
\frac{dm(t)}{dt} = \frac{d}{dt} \left[ 10 \cos(2\pi(2f_c)t) + 5 \cos(2\pi(3.6f_c)t) \right]
\]

\[ = 2\pi \left[ (-10 \times 2f_c) \sin(2\pi(2f_c)t) \right. \\
\left. + (-5 \times 3.6f_c) \sin(2\pi(3.6f_c)t) \right] \]

So that

\[
\left| \frac{dm(t)}{dt} \right| = 2\pi \left| (10 \times 2f_c) \sin(2\pi(2f_c)t) + (5 \times 3.6f_c) \sin(2\pi(3.6f_c)t) \right|
\]

The maximum of \( \left| \frac{dm(t)}{dt} \right| \) is achieved at either the maximum or minimum of the function \( g : IR \rightarrow IR \) given by

\[ g(t) = (10 \times 2f_c) \sin(2\pi(2f_c)t) + (5 \times 3.6f_c) \sin(2\pi(3.6f_c)t) \]
One could in principle compute $g(t)$ and solve (for $t$) the
equation $g(t)=0$. This approach, though correct, is quite
tedious as it does not yield a "neat" sol'n. Instead,
since $|\sin \theta| \leq 1$ for all $\theta \in \mathbb{R}$, we see that

$$\left| \frac{d\theta}{dt} \right| \leq 2\pi \left[ (10 \times 2f_c) + (5 \times 3f_c) \right], \quad t \in \mathbb{R}$$

so that

$$\left| \frac{d\theta}{dt} \right| \leq 70 \pi f_c, \quad t \in \mathbb{R}$$

The requirement on $\Delta$ and $T_s$ is now implied by

$$\frac{\Delta}{T_s} \geq 70 \pi f_c$$

i.e.,

$$\frac{\Delta}{T_s} \geq 70 \times \pi \times 250 = 17,500 \pi.$$
a) It is easy to check graphically that the needed condition
\[ H_{VSB}(f + fc) + H_{VSB}(f - fc) = 1, \quad |f| \leq W \]

b) Here
\[ B_T = \frac{W + \frac{W}{2}}{2} = \frac{3W}{2} \]

Note that
\[ y(t) = m(t)A_c \cos(2\pi f_mt) \]
\[ = A_c A_m \cos(2\pi f_mt) \cos(2\pi f_f t) \]
\[ = \frac{A_c A_m}{2} \left[ \cos(2\pi (f_m + fc)t) + \cos(2\pi (f_m - fc)t) \right] \]

Since \( f_m = cW, 0 < c < 1 \), and \( W < fc \), we have \( f_m < fc \) - Pictorially
With $\frac{1}{2} < c < 1$, \[ f_c + f_m = f_c + cW > f_c + \frac{1}{2}W \]
and
\[ f_c - f_m = f_c - cW < f_c - \frac{W}{2} \]

Therefore, direct inspection of $Y(f)$ and $H_{VSB}(f)$ shows that
\[ s_{VSB}(t) = \frac{A_c A_m}{2} \cos (2\pi (f_c + f_m)t), \quad t \in \mathbb{R}. \]

With $0 < c < \frac{1}{2}$,
\[ f_c - \frac{W}{2} < f_c - f_m < f_c + f_m < f_c + \frac{W}{2}. \]

Also,

Straight line on $|f - f_c| < \frac{W}{2}$:

\[ H_{VSB}(f) = \frac{1}{W} \left( f - (f_c - \frac{W}{2}) \right) + 0 \]

\[ = \frac{1}{W} \left( f - f_c + \frac{W}{2} \right) \]
At \( f = f_c - f_m = f_c - cW \),

\[
H_{VSB}(b) = \frac{1}{W} \left( f_c - cW - f_c + \frac{W}{2} \right) = \left( \frac{1}{2} - c \right).
\]

At \( f = f_c + f_m = f_c + cW \),

\[
H_{VSB}(b) = \frac{1}{W} \left( f_c + cW - f_c + \frac{W}{2} \right) = \frac{1}{2} + c.
\]

Direction inspection of \( Y(b) \) and \( H_{VSB}(b) \) now yields

\[
A_{VSB}(t) = \frac{A_c A_m}{2} \left[ \left( \frac{1}{2} + c \right) \cos(2\pi(cf_c + f_m)t) \\
+ \left( \frac{1}{2} - c \right) \cos(2\pi(cf_c - f_m)t) \right]
\]
We have
\[ \Delta t = A \left[ \sin(2\pi (f_c + f_a) t) - \sin(2\pi (f_c - f_a) t) \right] + B \cos(2\pi f_c t) \]
\[ = A \left[ \sin(2\pi f_c t) \cos(2\pi f_a t) + \sin(2\pi f_a t) \cos(2\pi f_c t) \right. \]
\[ \left. - \sin(2\pi f_c t) \cos(2\pi f_a t) + \sin(2\pi f_a t) \cos(2\pi f_c t) \right] + B \cos(2\pi f_c t) \]
\[ = \left[ 2A \sin(2\pi f_c t) + B \right] \cos(2\pi f_c t). \]

AM with \( m(t) = 2A \sin(2\pi f_c t) \).

We have
\[ \Delta t = (2A \sin(2\pi f_c t) + B) \cos(2\pi f_c t) \]
\[ = \Delta t \cos(2\pi f_c t) - \Delta t \sin(2\pi f_c t) \]
\[ = 2A \sin(2\pi f_c t) + B \quad \text{and} \quad \Delta t(t) = 0, \quad t \in \mathbb{R} \]
\[ \text{Both } \Delta t \text{ and } \Delta t \text{ are LP.} \]

It is now clear that \( \Delta t(t) = \sqrt{\Delta t(t)^2 + \Delta t(t)^2} = |2A \sin(2\pi f_c t) + B| \)
\text{envelope detector (applied to } s) \text{ yields } \Delta t. \text{ Thus, perfect reconstruction possible if } 2A \sin(2\pi f_c t) + B > 0, \quad t \in \mathbb{R} \text{ which happens if } -2A + B > 0 \quad \text{if } 2A \leq B \)