Throughout, we consider the information-bearing signal $m : \mathbb{R} \rightarrow \mathbb{R}$. Its Fourier transform is given by

$$M(f) := \int_{\mathbb{R}} m(t)e^{-j2\pi ft} dt, \quad f \in \mathbb{R}$$

**Frequency modulation**

The FM waveform $s_{FM} : \mathbb{R} \rightarrow \mathbb{R}$ associated with the information-bearing signal $m$ is given by

$$s_{FM}(t) = Acos(\theta_{FM}(t)), \quad t \in \mathbb{R}$$

with

$$\theta_{FM}(t) = 2\pi fc t + 2\pi k_F \int_{0}^{t} m(r)dr, \quad t \in \mathbb{R}$$

**Phase modulation**

The PM waveform $s_{PM} : \mathbb{R} \rightarrow \mathbb{R}$ associated with the information-bearing signal $m$ is given by

$$s_{PM}(t) = Acos(\theta_{PM}(t)), \quad t \in \mathbb{R}$$

with

$$\theta_{PM}(t) = 2\pi fc t + \pi k_P m(t), \quad t \in \mathbb{R}$$

**Single-tone modulating signals**
In an attempt to understand how the spectrum of angle-modulated signals is shaped by that of the modulating signal, we consider the simple case of a single-tone modulating signal $m : \mathbb{R} \rightarrow \mathbb{R}$, say
\[ m(t) = A_m \cos (2\pi f_m t), \quad t \in \mathbb{R} \]
with amplitude $A_m > 0$ and frequency $f_m > 0$. In that case, we note that
\begin{align*}
\theta_{FM}(t) &= 2\pi f_c t + 2\pi \int_0^t A_m \cos (2\pi f_m r) \, dr \\
&= 2\pi f_c t + 2\pi \frac{k_F A_m}{2\pi f_m} \sin (2\pi f_m t) \\
&= 2\pi f_c t + \frac{k_F A_m}{f_m} \sin (2\pi f_m t) \\
&= 2\pi f_c t + \beta \sin (2\pi f_m t)
\end{align*}
(1)
where
\[ \beta := \frac{\Delta f}{f_m}, \quad \Delta f := k_F A_m. \]

Next,
\begin{align*}
\cos (\theta_{FM}(t)) &= \cos (2\pi f_c t + \beta \sin (2\pi f_m t)) \\
&= \Re \left( e^{j2\pi f_c t} e^{j\beta \sin (2\pi f_m t)} \right)
\end{align*}
(2)
The function $t \rightarrow e^{j\beta \sin (2\pi f_m t)}$ being continuous and periodic with period $T_m = \frac{1}{f_m}$, it admits the Fourier series representation given by
\[ e^{j\beta \sin (2\pi f_m t)} = \sum_k c_k e^{j2\pi kf_m t}, \quad t \in \mathbb{R} \]
with
\[ c_k = \frac{1}{T_m} \int_{-T_m}^{T_m} e^{j\beta \sin (2\pi f_m t)} e^{-j2\pi kf_m t} \, dt, \quad k = 0, \pm 1, \pm 2, \ldots \]

Now fix $k = 0, \pm 1, \pm 2, \ldots$ Upon making the change of variable $x = 2\pi f_m t$, we get
\begin{align*}
c_k &= \frac{1}{T_m} \int_{-T_m}^{T_m} e^{j\beta \sin (2\pi f_m t)} e^{-j2\pi kf_m t} \, dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin (x) - kx)} \, dx \\
&= J_k(\beta)
\end{align*}
(3)
where
\[ J_k(\beta) := \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin(x) - kx)} dx, \quad \beta \in \mathbb{R} \]
defines the \( k \)th order Bessel function of the first kind.

Substituting we find
\[ e^{j\beta \sin(2\pi f_m t)} = \sum_k J_k(\beta) e^{j2\pi kf_m t}, \quad t \in \mathbb{R}. \]

Therefore,
\[ A_c \cos (\theta_{FM}(t)) = A_c \Re \left( e^{j2\pi f_c t} e^{j\beta \sin(2\pi f_m t)} \right) \]
\[ = A_c \Re \left( e^{j2\pi f_c t} \sum_k J_k(\beta) e^{j2\pi kf_m t} \right) \]
\[ = A_c \sum_k J_k(\beta) \Re \left( e^{j2\pi f_c t} e^{j2\pi kf_m t} \right) \]
\[ = A_c \sum_k J_k(\beta) \cos \left( 2\pi (f_c + k f_m) t \right). \]

(4)

In the frequency domain this last relationship becomes
\[ S_{FM}(f) = \frac{A_c}{2} \sum_k J_k(\beta) \left( \delta(f - (f_c + k f_m)) + \delta(f + (f_c + k f_m)) \right). \]

(5)

Thus, although the single-tone signal \( m \) has frequency content at the frequencies \( f = \pm f_m \), the corresponding FM wave has infinite bandwidth since it displays frequency content at the countably infinite set of frequencies
\[ f = \pm(f_c + k f_m), \quad k = 0, \pm 1, \ldots. \]

Carson’s formula

The realization that the spectrum of \( s_{FM} \) has infinite extent leads to the following practical concern: How much bandwidth is needed to transmit \( s_{FM} \) without too much distortion?

One answer to this question was given by Carson, and is summarized in the formula that carries his name: Carson’s formula states that the transmission bandwidth \( B_T \) of the FM wave associated with the single-tone signal \( m \) is well approximated by
\[ B_{T,\text{Carson}} := 2f_m + 2\Delta f \]
\[ = 2f_m (1 + \beta) \]

(6)
since $\Delta f = f_m \beta$ by definition.

One way to generalize Carson’s bandwidth formula could proceed formally by giving the quantities $f_m$ and $\beta$ interpretations which do not rely on the specific form of the information-bearing signal $m$. We do this as follows:

In the single-tone case, the frequency $f_m$ can be interpreted as the cutoff frequency of the signal – In other words, $f_m$ is the bandwidth of the signal. On the other hand, $\Delta f$ can be viewed as describing the largest possible excursion of the instantaneous frequency from $f_c$. Indeed, the instantaneous frequency of the FM wave at time $t$ is given by

$$\frac{1}{2\pi} \frac{d}{dt} \theta_{FM}(t) = f_c + k_F A_m \cos (2\pi f_m t)$$

and the corresponding deviation in instantaneous frequency at time $t$ is simply

$$\frac{1}{2\pi} \frac{d}{dt} \theta_{FM}(t) - f_c = k_F A_m \cos (2\pi f_m t).$$

Therefore, the maximal deviation from $f_c$ is given by

$$\sup (|k_F A_m \cos (2\pi f_m t)|, \quad t \in \mathbb{R}) = k_F A_m = \Delta f.$$ 

Now consider an information bearing signal which is bandlimited with cutoff frequency $W > 0$. With the discussion for the single-tone modulating signal in mind, it is natural to replace in Carson’s formula $f_m$ by $W$ and $\Delta f$ by

$$D := \sup (|k_F |m(t)|, \quad t \in \mathbb{R})$$

This suggests the approximation

$$B_T \simeq B_{T,\text{Carson}}$$

with

$$B_{T,\text{Carson}} := 2W + 2D$$

(7)

$$= 2W (1 + \beta)$$

where $\beta$ is defined as

$$\beta := \frac{D}{W} = \frac{\sup (|k_F |m(t)|, \quad t \in \mathbb{R})}{W}.$$
At this point, you may feel that the generalized Carson’s formula discussed above is simply a formal expression without much practical grounding. We now show through an approximation argument (see below) that the bandwidth as given by $B_{T,\text{Carson}}$ is indeed meaningful from an engineering point of view.

The basic idea is to characterize the spectrum of the FM wave associated with a sampled version of the information-bearing signal. Thus, fix $T > 0$. We approximate the information-bearing signal $m : \mathbb{R} \rightarrow \mathbb{R}$ by the staircase approximation $m_T^* : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$m_T^*(t) = m(kT), \quad kT \leq t < (k+1)T$$

with $k = 0, \pm 1, \ldots$. We then replace $\theta_{\text{FM}} : \mathbb{R} \rightarrow \mathbb{R}$ as defined above by $\theta_{\text{FM},T}^* : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$\theta_{\text{FM},T}^*(t) = 2\pi f_c t + 2\pi k_F \int_0^t m_T^*(r)dr, \quad t \in \mathbb{R}$$

and write

$$s_{\text{FM},T}^*(t) = A_c \cos (\theta_{T}^*(t)), \quad t \in \mathbb{R}.$$ 

Note that

$$S_{\text{FM},T}^*(f) = \int_{\mathbb{R}} A_c \cos (\theta_{T}^*(t)) e^{-j2\pi ft} dt$$

$$= A_c \sum_k \int_{kT}^{(k+1)T} \cos (\theta_{T}^*(t)) e^{-j2\pi ft} dt. \quad (8)$$

Now, for $k = 0, 1, \ldots$, with $kT \leq t < (k+1)T$, we have

$$\theta_{\text{FM},T}^*(t) = 2\pi f_c t + 2\pi k_F \int_0^t m_T^*(r)dr$$

$$= 2\pi f_c t + 2\pi k_F \left( T \sum_{\ell=0}^{k-1} m(\ell T) + m(kT)(t-kT) \right)$$

$$= 2\pi (f_c + k_F m(kT))(t-kT) + 2\pi T \left( k_F + k_F \sum_{\ell=0}^{k-1} m(\ell T) \right)$$

$$= 2\pi (f_c + k_F m(kT))(t-kT) + 2\pi \gamma_k T \quad (9)$$
where we have set
\[ \gamma_k := k f_c + kF \left( \sum_{\ell=0}^{k-1} m(\ell T) \right). \]

Therefore,
\[ s_{FM,T}^*(t) = A_c \cos \left( 2\pi \left( f_c + kF m(kT) \right)(t - kT) + 2\pi \gamma_k T \right) \]
and direct substitution yields
\[
\int_{kT}^{(k+1)T} \cos \left( \theta_T^*(t) \right) e^{-j2\pi f t} dt = A_c \int_{kT}^{(k+1)T} \cos \left( 2\pi \left( f_c + kF m(kT) \right)(t - kT) + 2\pi \gamma_k T \right) e^{-j2\pi f t} dt 
\]
(10)

To evaluate this last integral, we note that
\[
\int_0^T e^{\pm j2\pi((f_c+kFm(kT))\tau+\gamma_k T)} e^{-j2\pi f \tau} d\tau 
= e^{\pm j2\pi \gamma_k T} \int_0^T e^{j2\pi(\pm(f_c+kFm(kT))-f)\tau} d\tau 
= e^{\pm j2\pi \gamma_k T} \cdot \frac{e^{j2\pi(\pm(f_c+kFm(kT))-f)T} - 1}{j2\pi (\pm(f_c+kFm(kT))-f)} 
= a_k^\pm(f) \frac{\sin \left( \pi \left( \pm(f_c+kFm(kT))-f \right) T \right)}{\pi \left( \pm(f_c+kFm(kT))-f \right)} 
= a_k^\pm(f) \frac{\sin \left( \pi \left( f \mp(f_c+kFm(kT)) \right) T \right)}{\pi \left( f \mp(f_c+kFm(kT)) \right)} 
\]
(11)
with
\[ a_k^\pm(f) = e^{j2\pi \delta_k^\pm(f) T} \]
where
\[ \delta_k^\pm(f) = \pm \gamma_k + \frac{1}{2} \left( \pm(f_c+kFm(kT))-f \right). \]

Now recall that the sinc function \( \text{sinc} : \mathbb{R} \to \mathbb{R} \) is given by
\[ \text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}, \quad x \in \mathbb{R}. \]
Therefore, for each $k = 0, 1, \ldots$, we have

$$
\int_{kT}^{(k+1)T} \cos (\theta^*_T(t)) e^{-j2\pi f t} dt
= \frac{1}{2} a_k^+(f) \frac{\sin (\pi (f - (f_c + kFm(kT))) T)}{\pi (f - (f_c + kFm(kT)))}
+ \frac{1}{2} a_k^-(f) \frac{\sin (\pi (f + (f_c + kFm(kT))) T)}{\pi (f + (f_c + kFm(kT)))}
$$

(12)

Therefore,

$$
\int_0^\infty \cos (\theta^*_T(t)) e^{-j2\pi f t} dt
= \frac{1}{2} \sum_{k=0}^{\infty} a_k^+(f) \cdot \text{sinc} ((f - (f_c + kFm(kT))) T)
+ \frac{1}{2} \sum_{k=0}^{\infty} a_k^-(f) \cdot \text{sinc} ((f + (f_c + kFm(kT))) T)
$$

(13)

The zeroes of the sinc function occur at $x = \pm \ell, \ell = 1, 2, \ldots$, and its main lobe occupies the interval $[-1, 1]$. As a result, for each $k = 0, 1, \ldots$, the main contributions of the terms

$$
\frac{1}{2} a_k^+(f) \cdot \text{sinc} ((f - (f_c + kFm(kT))) T)
$$

is taking place on intervals centered at

$$
\pm(f_c + kFm(kT))
$$

and of length $2/T$, namely

$$
[\pm(f_c + kFm(kT)) - \frac{1}{T}, \pm(f_c + kFm(kT)) + \frac{1}{T}]
$$

Similar arguments could be made for the case $k = -1, -2, \ldots$ and would lead to a similar expression for

$$
\int_{-\infty}^0 \cos (\theta^*_T(t)) e^{-j2\pi f t} dt, \quad f \in \mathbb{R}.
$$
The discussion suggests that most of the spectral content is contained in the interval
\[ [\pm (f_c - D) - \frac{1}{T}, \pm (f_c + D) + \frac{1}{T}] \quad k = 0, \pm 1, \ldots \]
since
\[ |k_F m(kT)| \leq D, \quad k = 0, \pm 1, \ldots \]
by the definition of \( D \). This leads to estimating the transmission bandwidth of \( s_{FM,T}^* \) as being
\[ B_T \approx 2D + \frac{2}{T} \]
If we sample at the Nyquist rate, then \( T = \frac{1}{2W} \), and the information contained in \( m \) is recoverable from \( m_T^* \), and the transmission bandwidths of their corresponding FM waveforms should be commensurate. In short,
\[ B^* = 2D + 4W \]
is expected to provide a reasonably good approximation to \( B_T \). Note that
\[ B^* = 2D + 2W + 2W = B_{T,Carson} + 2W \]
so that this argument provides an approximation to the transmission bandwidth of the FM wave \( s_{FM} \) which is more conservative than the one provide by Carson’s formula. This can be traced to the fact that the approximation is based on a sampling argument.