Please work out the ten (10) problems stated below – BT refers to the text: D.P. Bertsekas and J.N. Tsitsiklis, Introduction to Probability (Second Edition), Athena Scientific (2008). Problem 1.55 (BT) refers to Problem 55 for Chapter 1 of BT (to be found at the end of Chapter 1). Show work and explain reasoning.

1. Consider a discrete rv $X : \Omega \to \mathbb{R}$ with support $S = \mathbb{N}$ such that $\mathbb{P} \left[ X = x \right] > 0$ for each $x = 0, 1, \ldots$

1.a. For each $t = 0, 1, \ldots$, compute the conditional probabilities

$$\mathbb{P} \left[ (X - t)^+ = x | X \geq t \right], \quad x = 0, 1, \ldots$$

1.b. Is it possible to find a pmf for the rv $X$ (with support $S = \mathbb{N}$) so that we simultaneously have

$$\mathbb{P} \left[ (X - t)^+ = x | X \geq t \right] = \mathbb{P} \left[ X = x \right], \quad x = 0, 1, \ldots$$

2. We start with two independent rvs $X, Y : \Omega \to \mathbb{R}$ which are discrete and have supports contained in $\mathbb{N}$. If $X$ is a Binomial rv Bin($n; a$) and $Y$ is a Binomial rv Bin($m; a$) for arbitrary positive integers $n$ and $m$ with $0 < a < 1$,

2.a. Find the pmf of the rv $X + Y$.

2.a. Evaluate the conditional expectation $\mathbb{E} \left[ X | X + Y = z \right]$ for each $z = 0, 1, \ldots, n + m$.

3. The rvs $X_1, \ldots, X_n : \Omega \to \mathbb{R}$ are mutually independent geometric rvs on $\mathbb{N}$ (not on $\mathbb{N}_0$), say

$$\mathbb{P} \left[ X_k = x \right] = a_k (1 - a_k)^x, \quad x = 0, 1, \ldots \quad k = 1, \ldots, n$$

with arbitrary parameters $0 < a_1, \ldots, a_n < 1$. 
3.a. Compute the probability
\[ P[X_1 = X_2 = \ldots = X_n]. \]

3.b. If \( a_1 = \ldots = a_n \equiv a \), what happens to \( P[X_1 = X_2 = \ldots = X_n] \) when \( n \) becomes large? Does it match your intuition?

4. Consider a discrete rv \( X : \Omega \to \mathbb{R} \) with support contained in \( \mathbb{N} \).
   
4.a. Show that its expectation \( \mathbb{E}[X] \) can also be computed by using the expression
   
   \[ \mathbb{E}[X] = \sum_{x=0}^{\infty} P[X > n]. \]

4.b. Use Part a to evaluate \( \mathbb{E}[X] \) when \( X \) has a geometric pmf on \( \mathbb{N} \) – The calculations are a lot simpler than the ones carried out using the direct approach!

5. Without calculations explain why the variance \( \text{Var}[X] \) of a Binomial rv \( \text{Bin}(n; p) \) is given by \( np(1 - p) \) with positive integer \( n \) and \( 0 < p < 1 \).

6. With rvs \( X_1, \ldots, X_n : \Omega \to \mathbb{R} \), we associate the rvs \( X^*, X_* : \Omega \to \mathbb{R} \) given by
   
   \[ X^* \equiv \max(X_1, \ldots, X_n) \quad \text{and} \quad X_* \equiv \min(X_1, \ldots, X_n). \]

   Assume that the rvs \( X_1, \ldots, X_n \) are discrete rvs which are independent and identically distributed with common pmf \( \{p(x), \ x \in S\} \) supported on the countable set \( S \subseteq \mathbb{R} \), i.e.,
   
   \[ P[X_i = x] = p(x), \quad x \in S \quad i = 1, \ldots, n. \]

6.a. Find the pmf of each of the discrete rvs \( X^* \) and \( X_* \).

6.b. Apply the results to Part a to the situation when the common pmf is the geometric pmf on \( \mathbb{N} \) given by
   
   \[ p(x) = a(1 - a)^x, \quad x = 0, 1, \ldots \]
   
   with \( 0 < a < 1 \). Do you notice anything interesting?

7. In this problem we consider evaluating \( \mathbb{E}[\frac{1}{1+X}] \) when the rv \( X : \Omega \to \mathbb{R} \) is a discrete rv with support contained in \( \mathbb{N} \). Do the calculations when
   
7.a. the rv \( X \) is a Binomial rv \( \text{Bin}(n; p) \) with positive integer \( n \) and \( 0 < p < 1 \).

7.b. the rv \( X \) is a Poison rv \( \text{Poi}(\lambda) \) with \( \lambda > 0 \).
8. Consider the following setting: The $n+1$ discrete rvs $X_1, \ldots, X_n, \nu : \Omega \rightarrow \mathbb{R}$ are mutually independent rvs. We shall assume that the rvs $X_1, \ldots, X_n$ are discrete rvs which are independent and identically distributed with common pmf $\{p(x), x \in S\}$ supported on the countable set $S \subseteq \mathbb{R}$, i.e.,

$$
P[X_i = x] = p(x), \quad x \in S, \quad i = 1, \ldots, n.
$$

Moreover, the rv $\nu$ is a discrete rv with support $\{0, 1, \ldots, N\}$ for some positive integer $N$. We shall assume that the (common) expectation of the rvs $X_1, \ldots, X_n$ is finite.

Show that that the expression

$$
E \left[ \sum_{k=1}^{\nu} X_i \right] = E[\nu] E[X_1]
$$

holds; this formula is known as Wald’s identity.

9. Let $X$ and $Y$ be two independent rvs which are uniformly distributed on the set of integers $\{0, \ldots, 9\}$, i.e.,

$$
P[X = x] = P[Y = y] = \frac{1}{10}, \quad x, y = 0, \ldots, 9.
$$

You are told that their sum $X + Y$ is of the form

$$
X + Y = \xi \cdot 10 + \eta
$$

where $\xi$ and $\eta$ are discrete rvs taking values in $\{0, \ldots, 9\}$, i.e.,

$$
P[\xi \in \{0, \ldots, 9\}] = P[\eta \in \{0, \ldots, 9\}] = 1.
$$

9.a. Explain how you would go about computing the probabilities

$$
P[\xi = x, \eta = y], \quad x, y = 0, \ldots, 9.
$$

9.b. Compute the probabilities

$$
P[\xi = 0, \eta = 0], \ P[\eta = 0] \text{ and } P[\xi = 0].
$$

What can you say about the independence of the rvs $\xi$ and $\eta$?

10. A point $X$ is picked uniformly at random from the set of integers $\{0, \ldots, \nu\}$ for some positive integer $\nu$ which is itself selected uniformly at random from the set $\{1, \ldots, N\}$ for some positive integer $N$. Thus,

$$
P[\nu = k] = \frac{1}{N}, \quad k = 1, 2, \ldots, N.
$$
10.a. Find the probabilities

\[ P[\nu = k | X < j], \quad k = 1, 2, \ldots, N \]

\[ j = 1, \ldots, N \]

10.b. Compute

\[ \mathbb{E}[\nu = k | X < j], \quad j = 1, \ldots, N \]

10.c. Compute \( \mathbb{E}[X] \).