ENGINEERING PROBABILITY

HOMEWORK # 13:
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Please work out the ten (10) problems stated below – BT refers to the text: D.P. Bertsekas and J.N. Tsitsiklis, Introduction to Probability (Second Edition), Athena Scientific (2008). Problem 1.55 (BT) refers to Problem 55 for Chapter 1 of BT (to be found at the end of Chapter 1). Show work and explain reasoning.

1. **A very important problem**\(^1\) – Let \(X\) and \(Y\) be independent rvs with \(X\) (resp. \(Y\)) normally distributed\(^2\) with mean \(a\) (resp. \(b\)) and variance \(\alpha^2 > 0\) (resp. \(\beta^2 > 0\)). Show by a completion of squares argument that the rv \(Z = X + Y\) is also normally distributed, and determine its mean and variance.

   Consider now the following situation: The \(n\) rvs \(X_1, \ldots, X_n\) are mutually independent rvs. For each \(i = 1, \ldots, n\), the rv \(X_i\) is normally distributed with mean \(a_i\) and variance \(\alpha_i^2 > 0\). Show that the rv \(X_1 + \ldots + X_n\) is also normally distributed, and identify its parameters.

2. Let \(X, Y\) and \(Z\) be independent and identically distributed rvs of the discrete type with

\[
\mathbb{P}[X = 1] = \mathbb{P}[X = 2] = \frac{1}{2}.
\]

2.a Find the pmf of the rv \(A = XYZ\).

2.b Find the pmf of the rv \(B = XY + YZ + ZX\).

3. Problems 4.17, 4.18 and 4.19 (BT)

4. Let \(X\) and \(Y\) be two independent, each normally distributed with zero mean and unit variance.

\(^1\)See also Problem 4.28 (BT)

\(^2\)A rv is normally distributed is just another way to say that it is a gaussian or normal rv.
4.a. Find the joint distribution of the rvs $U$ and $V$ where

$$U = X, \quad V = \begin{cases} \frac{Y}{X} & \text{if } X \neq 0 \\ 0 & \text{if } X = 0. \end{cases}$$

4.b. Determine whether this joint distribution is of continuous type. In the affirmative, identify the joint pdf.

4.c. Are the rvs $U$ and $V$ independent? Explain!

5. If $X$ and $Y$ are independent rvs which each uniformly distributed on $(0, 1)$, compute a closed form expression for the moment

$$E \left[ \frac{\max(X, Y)}{\min(X, Y)} 1[\min(X, Y) \neq 0] \right].$$

6. Consider two rvs $X$ and $Y$ which are independent and exponentially distributed, with parameter $\lambda > 0$ and $\mu > 0$, respectively. The rv $T$ is the function of $X$ and $Y$ given by

$$T = \begin{cases} \frac{Y}{X} & \text{if } X \neq 0 \\ 0 & \text{if } X = 0. \end{cases}$$

6.a. Compute the cumulative distribution function $F_T : \mathbb{R} \to [0, 1]$ of $T$, namely

$$F_T(t) = \mathbb{P}[T \leq t], \quad t \in \mathbb{R}$$

6.b. Show that the rv $T$ is of continuous type and identify its probability density function $f_T : \mathbb{R} \to \mathbb{R}_+$.

6.c. Compute the first moment $E[T]$. Carefully explain your answer!

7. With positive integer $n \geq 3$, let $R_1, R_2, \ldots, R_n$ be $n$ mutually independent and identically distributed rvs, each exponentially distributed with parameter $\lambda > 0$.

7.a. Explicitly compute the probability $\mathbb{P}[R_1 > 2R_2]$.

7.b. Explicitly compute the probability $\mathbb{P}[R_1 > 2R_2 > 3R_3]$.

7.c. Does the probability $\mathbb{P}[R_1 > 2R_2 > \ldots > (n-1)R_{n-1} > nR_n]$ depend on the parameter $\lambda$? Explain your answer!

7.d. Compute the probability $\mathbb{P}[R_1 > R_2 > \ldots > R_{n-1} > R_n]$.

8. Problem 4.22 (BT).
9. 
Problem 4.23 (BT).

10. 
Problem 4.24 (BT).