Please work out the ten (10) problems stated below – BT refers to the text: D.P. Bertsekas and J.N. Tsitsiklis, Introduction to Probability (Second Edition), Athena Scientific (2008). Problem 1.55 (BT) refers to Problem 55 for Chapter 1 of BT (to be found at the end of Chapter 1). Show work and explain reasoning.

1. The rvs $X$ and $Y$ are known to be jointly continuous with a probability density function $f_{X,Y}: \mathbb{R}^2 \rightarrow \mathbb{R}^+$ given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{e^{-y}}{y} & \text{if } 0 < x < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

1.a. Show that the rvs $X$ and $Y$ are each of continuous type, and identify their probability distribution functions $f_X, f_Y: \mathbb{R} \rightarrow \mathbb{R}$.

1.b. Are the rvs $X$ and $Y$ independent? Explain.

1.c. What is the conditional distribution of the rv $X$ given $Y = y$ for $y > 0$?

1.d. Is it easy to identify the conditional distribution of the rv $Y$ given $X = x$ for $x > 0$? Comments welcome!

1.e. Compute

$$\mathbb{E}[X^3|Y = y], \quad y > 0.$$ 

2. The rvs $X$ and $Y$ are known to be jointly continuous with a probability density function $f_{X,Y}: \mathbb{R}^2 \rightarrow \mathbb{R}^+$ given by

$$f_{X,Y}(x,y) = \begin{cases} e^{-\frac{x}{y}} e^{-y} & \text{if } 0 < x, y < \infty \\ 0 & \text{otherwise} \end{cases}$$
2.a. Show that the rvs \( X \) and \( Y \) are each of continuous type, and identify their probability distribution functions \( f_X, f_Y : \mathbb{R} \to \mathbb{R} \).

2.b. Are the rvs \( X \) and \( Y \) independent? Explain.

2.c. Compute
\[
\mathbb{E} [X^2 | Y = y], \quad y > 0.
\]

3. The rvs \( X \) and \( Y \) are jointly continuous with a probability density function \( f_{X,Y} : \mathbb{R}^2 \to \mathbb{R}_+ \) given by
\[
f_{X,Y}(x,y) = \begin{cases} 
\frac{x}{5} + cy & \text{if } 0 < x < 1, 1 < y < 5 \\
0 & \text{otherwise}
\end{cases}
\]
for some \( c > 0 \).

3.a. What is the value of \( c > 0 \)?
3.b. Are the rvs \( X \) and \( Y \) independent?
3.c. Evaluate \( \mathbb{P} [X + Y > 3] \). Recall that
\[
[X + Y > 3] = [(X, Y) \in B]
\]
where
\[
B = \{(x, y) \in \mathbb{R}^2 : x + y > 3 \}.
\]

4. The rvs \( X \) and \( Y \) are independent rvs with \( X \) (resp. \( Y \)) exponentially distributed with parameter 2 (resp. 3), respectively. With event \( A = [X + Y \leq 1] \), show that the joint conditional distribution of the pair \((X, Y)\) given \( A \) is of continuous type. Identify the joint conditional density function of the pair \((X, Y)\) given \( A \).

5. Problem 4.1 (BT)

6. Problem 4.2 (BT)

7. Problem 4.3 (BT)

8. Problem 4.4 (BT)

9. Problem 4.5 (BT)

10. Problem 4.6 (BT)