ENEE 324-01*/FALL 2018

ENGINEERING PROBABILITY

HOMEWORK # 10:
Posted on 04/04/2018

Please work out the ten (10) problems stated below – BT refers to the text: D.P. Bertsekas and J.N. Tsitsiklis, Introduction to Probability (Second Edition), Athena Scientific (2008). Problem 1.55 (BT) refers to Problem 55 for Chapter 1 of BT (to be found at the end of Chapter 1). Show work and explain reasoning.

1. If the rv $X : \Omega \rightarrow \mathbb{R}$ is a standard Gaussian rv, i.e., $X \sim N(0,1)$, show that the rv $Y : \Omega \rightarrow \mathbb{R}$ given by

$$Y \equiv \begin{cases} 
0 & \text{if } X \leq 0 \\
1 & \text{if } X > 0
\end{cases}$$

is a discrete rv and compute its pmf.

2. Problem 3.2 (BT)

3. Problem 3.5 (BT)

4. Problem 3.7 (BT)

5. Problem 3.11 (BT)

6. This problem should alert you to the centrality played by the standard normal distribution (i.e., $\mu = 0$ and $\sigma^2 = 1$): Problem 3.12 (BT)

7. Let $S$ be a bounded region of $\mathbb{R}^2$ with $\text{Area}(S) > 0$. The pair of rvs $X$ and $Y$ are said to be uniformly distributed on $S$ if

$$\mathbb{P} \left[(X,Y) \in B\right] = \int_B f(x,y)dxdy, \quad B \subseteq \mathbb{R}^2$$
where \( f : \mathbb{R} \rightarrow \mathbb{R}_+ \) is given by

\[
f(x, y) = \begin{cases} 
\frac{1}{\text{Area}(S)} & \text{if } (x, y) \in S \\
0 & \text{if } (x, y) \not\in S 
\end{cases}
\]

Show that the rvs \( X \) and \( Y \) are both (absolutely) continuous rvs, and identify their probability density functions \( f_X, f_Y : \mathbb{R} \rightarrow \mathbb{R}_+ \) in the following cases:

7.a. \( S \) is the unit square \([0, 1] \times [0, 1]\)

7.b. \( S \) is the diamond

\[
\{(x, y) \in [-1, 1] \times [-1, 1] : |y| \leq |x|\}.
\]

7.c. \( S \) is the unit disk

\[
\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.
\]

8. Problem 3.15 (BT)

9. The rv \( X : \Omega \rightarrow \mathbb{R} \) is of continuous type with probability density function \( f_X : \mathbb{R} \rightarrow \mathbb{R}_+ \) given by

\[
f_X(x) = \begin{cases} 
0 & \text{if } x < 1 \\
Cx^{-4} & \text{if } 1 \leq x 
\end{cases}
\]

for some constant \( C > 0 \).

9.a. What should be the value of \( C \)?

9.b. Find \( P([0.5 < X < 2]) \) and \( P([2 < X < 4]) \)

9.c. Find the CDF \( F_X : \mathbb{R} \rightarrow [01,] \) of the rv \( X \).

9.d. Find \( E[X^n] \) for all \( n = 1, 2, \ldots \)

10. The rv \( X : \Omega \rightarrow \mathbb{R} \) is of continuous type with probability density function \( f_X : \mathbb{R} \rightarrow \mathbb{R}_+ \) given by

\[
f_X(x) = \begin{cases} 
0 & \text{if } x < 1 \\
\frac{1}{2}x^{-\frac{3}{2}} & \text{if } 1 \leq x 
\end{cases}
\]

10.a. Find \( P(X > 10] \)

10.b. Find the CDF \( F_X : \mathbb{R} \rightarrow [01,] \) of the rv \( X \).

10.c. Find \( E[X] \).

10.d. Find \( E\left[X^{\frac{1}{2}}\right] \).