1. (OW 3.47) Consider the signal

\[ x(t) = \cos(2\pi t) \]

Since \( x(t) \) is periodic with a fundamental period of 1, it is also periodic with a period of \( N \), where \( N \) is any positive integer.

(a) What are the Fourier series coefficients, \( a_k \), if we regard \( x(t) \) as periodic with period 1.

\[
a_k = \frac{1}{T} \int_0^T x(t) e^{-j \frac{2\pi}{T} kt} dt
\]

\[
= \int_0^1 \frac{1}{2} \left[ e^{j2\pi t} + e^{-j2\pi t} \right] e^{-j \frac{2\pi}{T} kt} dt
\]

\[
a_k = \frac{1}{2} \int_0^1 e^{-j2\pi (k-1)t} dt + \frac{1}{2} \int_0^1 e^{-j2\pi (k+1)t} dt
\]

If \( k \neq \pm 1 \) we are integrating a complex exponential over an integer number of periods, so the result is zero.

\[
a_1 = \frac{1}{2} \int_0^1 e^{-j2\pi (1-1)t} dt = \frac{1}{2}
\]

\[
a_{-1} = \frac{1}{2} \int_0^1 e^{-j2\pi (-1+1)t} dt = \frac{1}{2}
\]

(b) What are the Fourier series coefficients, \( b_k \), if we regard \( x(t) \) as periodic with period 3.

\[
b_k = \frac{1}{T} \int_0^T x(t) e^{-j \frac{2\pi}{3} kt} dt
\]

\[
= \frac{1}{3} \int_0^3 \frac{1}{2} \left[ e^{j2\pi t} + e^{-j2\pi t} \right] e^{-j \frac{2\pi}{3} kt} dt
\]

\[
b_k = \frac{1}{6} \int_0^3 e^{-j(k-3)\frac{2\pi}{3} t} dt + \frac{1}{2} \int_0^3 e^{-j(k+3)\frac{2\pi}{3} t} dt
\]

If \( k \neq \pm 3 \) we are integrating a complex exponential over an integer number of periods, so the result is zero.

\[
b_3 = \frac{1}{6} \int_0^3 e^{-j2\pi (3-3)t} dt = \frac{1}{2}
\]

\[
b_{-3} = \frac{1}{6} \int_0^3 e^{-j2\pi (-3+3)t} dt = \frac{1}{2}
\]

Note that both (a) and (b) could have been done by inspection.
2. (OW 3.33) Consider a causal continuous-time LTI system whose input $x(t)$ and output $y(t)$ are related by the following differential equation:

$$\frac{d}{dt}y(t) + 4y(t) = x(t)$$

Find the Fourier series representation of the output $y(t)$ if the input signal is

$$x(t) = \cos(2\pi t)$$

To determine the frequency response of the system, recall that if the system input is given by $x(t) = e^{j\omega t}$, then the output is $y(t) = H(j\omega)e^{j\omega t}$. Substituting these into the differential equation we have

$$\frac{d}{dt} \left[ H(j\omega)e^{j\omega t} \right] + 4H(j\omega)e^{j\omega t} = e^{j\omega t}$$

$$H(j\omega)j\omega e^{j\omega t} + 4H(j\omega)e^{j\omega t} = e^{j\omega t}$$

$$H(j\omega)[4 + j\omega] e^{j\omega t} = e^{j\omega t}$$

Hence the frequency response of the system and the Fourier series of the output corresponding to an arbitrary periodic input signal are given by

$$H(j\omega) = \frac{1}{4 + j\omega}$$

$$y(t) = \sum_{k=-\infty}^{\infty} a_kH(j\omega_0k)e^{j\omega_0kt}$$

where $x(t)$ has Fourier series coefficients $a_k$ and fundamental frequency $\omega_0$. For $x(t) = \cos(2\pi t)$ we have $\omega_0 = 2\pi$ and the only nonzero Fourier series coefficients of $x(t)$ are $a_{-1} = a_1 = 1/2$. Therefore, the nonzero Fourier series coefficients for $y(t)$ are

$$b_1 = a_1H(j2\pi) = \frac{1}{2(4 + j2\pi)}$$

$$b_{-1} = a_{-1}H(-j2\pi) = \frac{1}{2(4 - j2\pi)}$$
3. (OW 3.34) Consider a continuous-time LTI system with impulse response

\[ h(t) = e^{-4|t|}. \]

Find the Fourier series coefficients of the output \( y(t) \) if the input signal is

\[ x(t) = \sum_{n=-\infty}^{\infty} \delta(t - n) \]

The frequency response of the system is

\[ H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt \]

\[ = \int_{-\infty}^{\infty} e^{-4|t|} e^{-j\omega t} dt \]

\[ = \int_{-\infty}^{0} e^{(4-j\omega)t} dt + \int_{0}^{\infty} e^{-(4+j\omega)t} dt \]

\[ = \frac{1}{4-j\omega} + \frac{1}{4+j\omega} \]

\[ H(j\omega) = \frac{8}{16 + \omega^2} \]

For the input signal given above, the period is \( T = 1 \), so \( \omega_0 = 2\pi \) and

\[ a_k = \frac{1}{T} \int_{T} x(t) e^{-j\frac{2\pi}{T}kt} dt = \int_{0}^{0.5} \delta(t) e^{-j2\pi kt} dt = e^{-j2\pi k(0)} = 1 \]

\[ b_k = H(j\omega_0 k) a_k = H(j2\pi k) = \frac{8}{16 + (2\pi k)^2} = \frac{2}{4 + (\pi k)^2} \]
4. (OW 3.36) Consider a causal discrete-time LTI system whose input $x[n]$ and output $y[n]$ are related by the following difference equation:

$$y[n] - \frac{1}{4}y[n - 1] = x[n]$$

Find the Fourier series representation of the output $y[n]$ if the input signal is

$$x[n] = \cos\left(\frac{\pi}{4}n\right) + 2\cos\left(\frac{\pi}{2}n\right)$$

To determine the frequency response of the system, recall that if the system input is given by $x[n] = e^{j\omega n}$, then the output is $y[n] = H(e^{j\omega})e^{j\omega n}$. Substituting these in the equation above,

$$y[n] - \frac{1}{4}y[n - 1] = x[n]$$

$$H(e^{j\omega})e^{j\omega n} - \frac{1}{4}H(e^{j\omega})e^{j\omega(n-1)} = e^{j\omega n}$$

$$H\left(e^{j\omega}\right)\left[1 - \frac{1}{4}e^{-j\omega}\right]e^{j\omega n} = e^{j\omega n}$$

Hence the frequency response of the system and the Fourier series of the output corresponding to an arbitrary periodic input signal are given by

$$H\left(e^{j\omega}\right) = \frac{1}{1 - \frac{1}{4}e^{-j\omega}}$$

$$y[n] = \sum_{k=\langle N\rangle} a_k H\left(e^{j2\pi k/N}\right) e^{j2\pi k/Nn}$$

where $x[n]$ has Fourier series coefficients $a_k$ and fundamental frequency $\omega_0$. For

$$x[n] = \cos\left(\frac{2\pi}{8}n\right) + 2\cos\left(\frac{2\pi}{8}n\right) = \frac{1}{2} e^{j\frac{2\pi}{8}n} + \frac{1}{2} e^{-j\frac{2\pi}{8}n} + e^{j\frac{2\pi}{8}n} + e^{-j\frac{2\pi}{8}n}$$

we have $N = 8$ and the nonzero Fourier series coefficients of $x[n]$ are $a_1 = a_{-1} = 1/2$ and $a_2 = a_{-2} = 1$. Therefore, the nonzero Fourier series coefficients for $y[n]$ are

$$b_1 = a_1 H\left(e^{j\pi/4}\right) = \frac{1}{2 \left(1 - \frac{1}{4}e^{-j\pi/4}\right)}$$

$$b_{-1} = a_{-1} H\left(e^{-j\pi/4}\right) = \frac{1}{2 \left(1 - \frac{1}{4}e^{j\pi/4}\right)}$$

$$b_2 = a_2 H\left(e^{j\pi/2}\right) = \frac{1}{1 - \frac{1}{4}e^{-j\pi/2}}$$

$$b_{-2} = a_{-2} H\left(e^{-j\pi/2}\right) = \frac{1}{1 - \frac{1}{4}e^{j\pi/2}}$$
5. Find the Fourier series representation of the output $y(t)$ for the system in problem 2 if the input signal is

$$x(t) = \sin(4\pi t) + \cos(6\pi t + \pi/4)$$

Here $\omega_0 = 2\pi$ and the nonzero FS coefficients of $x(t)$ are $a_2 = a^*_2 = \frac{1}{\sqrt{2}}$ and $a_3 = a^*_3 = \frac{1}{2}e^{j\pi/4}$. Therefore, the nonzero FS coefficients for $y(t)$ are

\[
\begin{align*}
  b_2 &= a_2 H(j4\pi) = \frac{1}{j2(4 + j4\pi)} \\
  b_{-2} &= a_{-2} H(-j4\pi) = \frac{-1}{j2(4 - j4\pi)} \\
  b_3 &= a_3 H(j6\pi) = \frac{\exp(j\pi/4)}{2(4 + j6\pi)} \\
  b_{-3} &= a_{-3} H(-j6\pi) = \frac{-\exp(j\pi/4)}{2(4 - j6\pi)}
\end{align*}
\]

6. Find the Fourier series representation of the output $y[n]$ for the system in problem 4 if the input signal is

$$x[n] = \sin\left(\frac{3\pi}{4} n\right)$$

Note that

$$x[n] = \sin\left(\frac{3\pi}{8} n\right) = \frac{1}{j2} e^{j\frac{3\pi}{4}n} - \frac{1}{j2} e^{-j\frac{3\pi}{4}n}$$

Here $N = 8$ and the only nonzero FS coefficients of $x[n]$ are $a_3 = a^*_{-3} = 1/j2$. Therefore, the nonzero FS coefficients for $y[n]$ are

\[
\begin{align*}
  b_3 &= a_3 H(e^{j3\pi/4}) = \frac{1}{j2 \left(1 - \frac{1}{4} e^{-j3\pi/4}\right)} \\
  b_{-3} &= a_{-3} H(e^{-j3\pi/4}) = \frac{-1}{j2 \left(1 - \frac{1}{4} e^{j3\pi/4}\right)}
\end{align*}
\]
7. (OW 3.38) Consider a discrete-time LTI system with impulse response \( h[n] \) and input \( x[n] \) given below

\[
h[n] = \begin{cases} 
-1, & -2 \leq n \leq -1 \\
1, & 0 \leq n \leq 2 \\
0, & \text{otherwise}
\end{cases}
\]

\[
x[n] = \sum_{k=-\infty}^{\infty} \delta[n - 4k]
\]

Determine the Fourier series coefficients of the output \( y[n] \).

The frequency response of the system is may be evaluated as

\[
H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}
\]

\[
= e^{-j2\omega} + e^{-j\omega} + 1 - e^{j\omega} - e^{j2\omega}
\]

\[
H(e^{j\omega}) = 1 - j2 \sin(\omega) - j2 \sin(2\omega)
\]

The input signal has period \( N = 4 \) so \( \omega_0 = 2\pi/4 = \pi/2 \) and FS coefficients

\[
a_k = \frac{1}{N} \sum_{n=-2}^{1} x[n]e^{-j2\pi n/N} = \frac{1}{4} \sum_{n=-2}^{1} \delta[n]e^{-j\pi n/2} = \frac{1}{4}
\]

Therefore, the output signal has FS coefficients

\[
b_k = a_k H(e^{j\pi k/2})
\]

\[
= a_k [1 - j2 \sin (\pi k/2) - j2 \sin (\pi k)]
\]

\[
= a_k [1 - j2 \sin (\pi k/2)]
\]

\[
b_k = \begin{cases} 
1, & k \text{ even} \\
1 - j2(-1)^{(k-1)/2}, & k \text{ odd}
\end{cases}
\]
8. (OW 3.39) Consider a discrete-time LTI system with frequency response

\[
H(e^{j\omega}) = \begin{cases} 
1, & |\omega| \leq \frac{\pi}{8} \\
0, & \frac{\pi}{8} < |\omega| < \pi 
\end{cases}
\]

Show that if the input, \( x[n] \), to the system has period \( N = 5 \), the output, \( y[n] \), has only one nonzero Fourier series coefficient per period.

Let the FS coefficients of the input be \( a_k \), then the FS coefficients of the output are

\[
b_k = a_k H(e^{j2\pi k/N}) = a_k H(e^{j\frac{2\pi k}{5}})
\]

for \( k = -2, -1, 0, 1, 2 \). Note that

\[
2\pi/5 = \frac{16\pi}{40} > \frac{5\pi}{40} = \frac{\pi}{8} \\
4\pi/5 = \frac{32\pi}{40} > \frac{5\pi}{40} = \frac{\pi}{8}
\]

so for the frequency response given above we have

\[
\begin{align*}
H(e^{-j4\pi/5}) &= 0 \implies b_{-2} = a_{-2} H(e^{-j4\pi/5}) = 0 \\
H(e^{-j2\pi/5}) &= 0 \implies b_{-1} = a_{-1} H(e^{-j2\pi/5}) = 0 \\
H(e^{0}) &= 1 \implies b_1 = a_1 H(e^{0}) = a_1 \\
H(e^{j2\pi/5}) &= 0 \implies b_1 = a_1 H(e^{j2\pi/5}) = 0 \\
H(e^{j4\pi/5}) &= 0 \implies b_1 = a_1 H(e^{j4\pi/5}) = 0
\end{align*}
\]

Therefore, only \( b_0 \) can have a nonzero value for \( b_k \) in the range \(-2 \leq k \leq 2\).