1. (OW 1.30) Determine if each of the following systems is invertible. If it is, construct the inverse system. If it is not, find two input signals which that have the same output.

(a) \( y(t) = \int_{-\infty}^{t} x(\tau) d\tau \)

Invertible:
\[
z(t) = \frac{d}{dt}y(t) = \frac{d}{dt} \int_{-\infty}^{t} x(\tau) d\tau = x(t)
\]

(b) \( y(t) = x(2t - 4) \)

Invertible:
\[
z(t) = y \left( \frac{1}{2}t + 2 \right) = x \left( 2 \left( \frac{1}{2}t + 2 \right) - 4 \right) = x(t + 4 - 4) = x(t)
\]

(c) \( y(t) = x^n(t) \) where \( n \) is an integer.

- For even \( n \), the system is noninvertible because \( x_1(t) = u(t) \) and \( x_2(t) = -u(t) \) both produce the same output signal \( y(t) = u(t) \).

- For odd \( n \), the system is invertible with inverse system given by
\[
z(t) = \left[ y(t) \right]^{1/n} = \left[ x^n(t) \right]^{1/n} = x(t)
\]

(d) \( y[n] = nx[n] \)

noninvertible. \( \delta[n] \) and \( 2\delta[n] \) give the same output.

(e) \( y[n] = x[n]x[n - 1] \)

noninvertible. Note that \( x_1[n] = u[n] \) and \( x_2[n] = -u[n] \) both go to \( y_1[n] = y_2[n] = u[n - 1] \).
2. Conolve the following discrete-time signals

(a)

(b)

<table>
<thead>
<tr>
<th></th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x[0] \cdot h[n]$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x[1] \cdot h[n-1]$</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>2</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x[2] \cdot h[n-2]$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>6</td>
<td>3</td>
<td>-3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x[3] \cdot h[n-3]$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$y[n]$</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>8</td>
<td>8</td>
<td>3</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
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<th>4</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$x[-2] \cdot h[n+2]$</td>
<td>0</td>
<td>2</td>
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<td>2</td>
<td>0</td>
<td>0</td>
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</tr>
<tr>
<td>$x[-1] \cdot h[n+1]$</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>4</td>
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<td>0</td>
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</tr>
<tr>
<td>$x[0] \cdot h[n]$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
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<td>0</td>
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</tr>
<tr>
<td>$x[1] \cdot h[n-1]$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x[2] \cdot h[n-2]$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$y[n]$</td>
<td>0</td>
<td>2</td>
<td>6</td>
<td>12</td>
<td>14</td>
<td>12</td>
<td>6</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>
3. The impulse response of the RC circuit shown below is given by

\[ h(t) = \frac{1}{RC} \exp \left( -\frac{t}{RC} \right) u(t) \]

Determine the output if \( x(t) = \cos (\omega_0 t) \). Simplify your answer as much as possible.

\[
y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau \\
= \int_{-\infty}^{\infty} \cos (\omega_0 \tau) \frac{1}{RC} \exp \left( -\frac{t - \tau}{RC} \right) u(t - \tau) d\tau \\
= \frac{1}{RC} \exp \left( -\frac{t}{RC} \right) \int_{-\infty}^{t} \cos (\omega_0 \tau) \exp \left( -\frac{\tau}{RC} \right) d\tau \\
= \frac{1}{2RC} \exp \left( -\frac{t}{RC} \right) \left[ \exp \left( \frac{i\omega_0 t}{RC} + j\omega_0 \right) \right. \\
+ \left. \exp \left( \frac{i\omega_0 t}{RC} - j\omega_0 \right) \right] \\
= \frac{1}{2RC} \left[ \exp \left( j\omega_0 t \right) \right. \\
+ \left. \exp \left( -j\omega_0 t \right) \right] \\
= \frac{1}{2RC} \left[ \exp \left( j\omega_0 t \right) \right. \\
+ \left. \exp \left( -j\omega_0 t \right) \right] \\
= \frac{1}{2RC} \left[ \exp \left( j\omega_0 t \right) \right. \\
+ \left. \exp \left( -j\omega_0 t \right) \right] \\
= \frac{1}{2RC} \left[ \exp \left( j\omega_0 t - \arctan (\omega_0 / RC) \right) \right. \\
+ \left. \exp \left( -j\omega_0 t - \arctan (-\omega_0 / RC) \right) \right] \\
= \frac{1}{2RC} \left[ \frac{\exp \left( j\omega_0 t \right)}{\sqrt{1 + (\omega_0 / RC)^2}} + \frac{\exp \left( -j\omega_0 t \right)}{\sqrt{1 + (\omega_0 / RC)^2}} \right] \\
= \frac{1}{2RC} \frac{1}{\sqrt{1 + (\omega_0 / RC)^2}} \\
y(t) = A \cos (\omega_0 t + \theta)
\]

where we have defined

\[ A = \frac{1/(RC)}{\sqrt{\omega_0^2 + 1/(RC)^2}} \]
\[ \theta = \arctan (\omega_0 / RC) \]

Note that the output is sinusoidal signal which has the same frequency as the input. The system only changed the amplitude and phase.
4. Perform the following convolutions for the signals shown below.

(a) \( g(t) = x(t) \ast y(t) \)

**Region 1** For \( t + 1 \leq -1 \) there is no overlap, so \( y(t) = 0 \) for \( t \leq -2 \).

**Region 2** For \( t + 1 \geq -1 \) and \( t - 1 \leq -1 \)

\[
g(t) = \int_{t-1}^{t+1} d\tau = (t + 1) - (-1) = t + 2
\]

Hence, \( g(t) = t + 2 \) for \(-2 \leq t \leq 0\).

**Region 3** For \( t + 1 \geq 1 \) and \( t - 1 \leq 1 \)

\[
g(t) = \int_{t-1}^{1} d\tau = (1) - (t - 1) = 2 - t
\]

Hence, \( g(t) = 2 - t \) for \( 0 \leq t \leq 2 \).

**Region 4** For \( t - 1 \geq 1 \) there is no overlap, so \( y(t) = 0 \) for \( t \geq 2 \).

\[
g(t) = \begin{cases} 
0 & t \leq -2 \\
 t + 2 & -2 \leq t \leq 0 \\
2 - t & 0 \leq t \leq 2 \\
0 & t \geq 2 
\end{cases}
\]
(b) $h(t) = x(t) * z(t)$

**Region 1** For $t + 1 \leq 1$ there is no overlap, so $h(t) = 0$ for $t \leq 0$.

**Region 2** For $t - 1 \leq 1$ and $t + 1 \geq 1$

$$h(t) = \int_1^{t+1} d\tau = (t+1) - 1 = t$$

Hence $h(t) = t$ for $0 < t < 2$.

**Region 3** For $t - 1 \leq 3$ and $t + 1 \geq 3$

$$h(t) = \int_{t-1}^{3} d\tau = 3 - (t-1) = 4 - t$$

Hence $h(t) = 4 - t$ for $2 < t < 4$.

**Region 4** For $t - 1 > 3$ there is no overlap, so $h(t) = 0$ for $t > 40$.

$$h(t) = \begin{cases} 
0 & t \leq 0 \\
t & 0 \leq t \leq 2 \\
4 - t & 2 \leq t \leq 4 \\
0 & t \geq 4
\end{cases}$$

Note from parts (a) and (b) that $z(t) = y(t - 2)$ and $h(t) = g(t - 2)$, so shifting one signal shifts the result by the same amount (this follows directly from time invariance). Therefore, $x(t) * y(t - 1) = g(t - 1)$ and we have

$$r(t) = 2x(t + 2) + g(t - 1) + 2x(t - 2)$$

(c) $r(t) = x(t) * w(t)$

$$r(t) = x(t) * [2\delta(t + 2) + y(t - 1) + 2\delta(t - 3)] = 2x(t + 2) + x(t) * y(t - 1) + 2x(t - 2)$$
Additional Problems. Do Not Turn In!!

5. (OW 1.27 and 1.28) Systems can be characterized by a number of general properties:

(1) Memoryless vs. Memory
(2) Time invariant vs. Time varying
(3) Linear vs. nonlinear
(4) Causal vs. noncausal
(5) Stable vs. unstable

Determine which properties hold and which do not for each of the following systems. You must justify your answers. In each example, \( y(t) \) or \( y[n] \) denotes the system output and \( x(t) \) or \( x[n] \) is the system input.

(a) \( y(t) = \frac{d}{dt}x(t) \)

(1,4) Note that

\[
\frac{d}{dt} = \lim_{\Delta \to \infty} \frac{x(t + \Delta) - x(t - \Delta)}{2\Delta}
\]

when considering the derivative at a specific point we are looking at a time a little bit before and a little bit after \( t \). Since \( y(t) \) depends on past and future values of \( x(t) \), the system has **Memory** and is **Noncausal**.

(2) For time invariance, consider

\[
\begin{align*}
  y_1(t) &= \frac{d}{dt}x_1(t) \\
  y_1(t - t_0) &= \frac{d}{dt}x_1(t - t_0) \\
  x_2(t) &= x_1(t - t_0) \\
  y_2(t) &= \frac{d}{dt}x_2(t) = \frac{d}{dt}x_1(t - t_0) \\
  y_1(t - t_0) &= y_2(t)
\end{align*}
\]

The system is **Time Invariant**

(3) Let \( x_3(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t) \) Then

\[
\begin{align*}
  y_3(t) &= \frac{d}{dt}x_3(t) \\
  &= \frac{d}{dt} [\alpha_1 x_1(t) + \alpha_2 x_2(t)] \\
  &= \alpha_1 \left[ \frac{d}{dt} x_1(t) \right] + \alpha_2 \left[ \frac{d}{dt} x_2(t) \right] \\
  &= \alpha_1 y_1(t) + \alpha_2 y_2(t)
\end{align*}
\]

Hence the system is **Linear**.
(5) Consider the input signal \( u_\triangle(t) \) defined as

\[
u_\triangle(t) = \begin{cases} \frac{1}{\Delta} t & 0 \leq t \leq \Delta \\ 1 & \text{elsewhere} \end{cases}
\]

The output is

\[
y_\triangle(t) = \frac{d}{dt} y_\triangle(t) = \begin{cases} \frac{1}{\Delta} & 0 \leq t \leq \Delta \\ 0 & \text{elsewhere} \end{cases}
\]

In the limit as \( \Delta \) goes to zero, the input becomes a unit step function which is bounded and the output becomes unbounded; hence the system is \textbf{unstable}.

(b) \( y[n] = (1 + (-1)^n) x[n] \)

(1,4) \( y[n] \) only depends on present values of \( n \) so the system is \textbf{Memoryless} and \textbf{Non-causal}.

(2) Let \( x_2[n] = x_1[n - n_0] \)

\[
y_1[n - n_0] = (1 + (-1)^{n-n_0}) x[n - n_0]
x_2[n] = x_1[n - n_0]
y_2[n] = (1 + (-1)^n) x_2[n] = (1 + (-1)^n) x_1[n - n_0]
\]

Since \( y_1[n - n_0] \neq y_2[n] \), the system is \textbf{Time Varying}

(3) Let \( x_3[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n] \)

\[
y_3[n] = (1 + (-1)^n) x_3[n - n_0]
= (1 + (-1)^n) (\alpha_1 x_1[n] + \alpha_2 x_2[n])
= \alpha_1 (1 + (-1)^n) x_1[n] + \alpha_2 (1 + (-1)^n) x_2[n]
= \alpha_1 y_1[n] + \alpha_2 y_2[n]
\]

(5) If \( |x[n]| \leq B \) for all \( n \), then

\[
|y[n]| = |(1 + (-1)^n) x[n]| \leq |1| + |(-1^n)| |1 + (-1)^n| \cdot |x[n]| \leq 2B
\]

Hence the system is \textbf{BIBO Stable}

(c) \( y(t) = x(t - 2) + x(2 - t) \)

(1) \( y(0) \) depends on \( x(t) \) at time \( t = -2 \). Therefore, the system has memory.

(2) Let \( x_1(t) = x(t - T) \), then

\[
y_1(t) = x_1(t - 2) + x_1(2 - t)
= x(t - 2 - T) + x(2 - t - T)
y_1(t) = x((t - T) - 2) + x(2 - (t + T))
\]

Note that

\[
y(t) = x(t - 2) + x(2 - t)
y(t - T) = x((t - T) - 2) + x(2 - (t - T))
\]

Since \( y_1(t) \neq y(t - T) \), the system is time-varying.
(3) Assume \( x_1(t) \to y_1(t) \) and \( x_2(t) \to y_2(t) \) and \( x_3(t) = ax_1(t) + bx_2(t) \)

\[
y_3(t) = x_3(t - 2) + x_3(2 - t) \\
= [ax_1(t - 2) + bx_2(t - 2)] + [ax_1(2 - t) + bx_2(2 - t)] \\
= a[x_1(t - 2) + x_1(2 - t)] + b[x_2(t - 2) + x_2(2 - t)] \\
y_3(t) = ay_1(t) + by_2(t)
\]

Hence the system is linear.

(4) \( y(0) \) depends on \( x(t) \) at time \( t = 2 \). Therefore, the system is noncausal.

(5) If \( |x(t)| \leq M_x \) for all \( t \), then

\[
|y(t)| = |x(t - 2) + x(2 - t)| \leq |x(t - 2)| + |x(2 - t)| \leq M_x + M_x = 2M_x
\]

Hence the system is BIBO stable.

(d) \( y(t) = \int_{-\infty}^{2t} x(\tau) d\tau \)

(1) The system has memory.

(2) Let \( x_1(t) = x(t - T) \), then

\[
y_1(t) = \int_{-\infty}^{2t} x_1(\tau) d\tau = \int_{-\infty}^{2t} x(\tau - T) d\tau = \int_{-\infty}^{2t-T} x(\lambda) d\lambda \\
y(t - T) = \int_{-\infty}^{2(t-T)} x(\tau) d\tau = \int_{-\infty}^{2t-2T} x(\tau) d\tau
\]

Since \( y_1(t) \neq y(t - T) \), the system is time-varying.

(3) Assume \( x_1(t) \to y_1(t) \) and \( x_2(t) \to y_2(t) \) and \( x_3(t) = ax_1(t) + bx_2(t) \)

\[
y_3(t) = \int_{-\infty}^{2t} x_3(\tau) d\tau \\
= \int_{-\infty}^{2t} [ax_1(\tau) + bx_2(\tau)] d\tau \\
= a\int_{-\infty}^{2t} x_1(\tau) d\tau + b\int_{-\infty}^{2t} x_2(\tau) d\tau \\
y_3(t) = ay_1(t) + by_2(t)
\]

Hence the system is linear.

(4) \( y(t) \) at time \( t = 2 \) depends on \( x(t) \) at time \( t = 4 \) so the system is noncausal.

(5) Consider the bounded input signal \( x(t) = u(t) \).

\[
y(t) = \int_{-\infty}^{2t} x(\tau) d\tau = \int_{0}^{2t} d\tau = 2t u(t)
\]

This blows up as \( t \to \infty \), so the system is not BIBO stable.

(e) \( y[n] = x[-n] \)

(1) \( y[1] \) depends on \( x[-1] \) so the system has memory.
(2) Let \( x_1[n] = x[n - N] \), then
\[
\begin{align*}
y_1[n] &= x_1[-n] = x[-n - N] \\
y[n - N] &= x[-(n - N)] = x[-n + N]
\end{align*}
\]

Hence the system is time-varying.

(3) Assume \( x_1[n] \rightarrow y_1[n] \) and \( x_2[n] \rightarrow y_2[n] \) and \( x_3[n] = ax_1[n] + bx_2[n] \)
\[
y_3[n] = x_3[-n] = ax_1[-n] + bx_2[-n] = ay_1[n] + by_2[n]
\]

Hence the system is linear.

(4) \( y[-1] \) depends on \( x[1] \) so the system is noncausal.

(5) If \( |x[n]| \leq M_x \) for all \( n \), then \( |y[n]| = |x[-n]| \leq M_x \) for all \( n \). Therefore, the system is BIBO stable.

\[ y[n] = \begin{cases} 
  x[n], & n \geq 1 \\
  0, & n = 0 \\
  x[n + 1], & n \leq -1 
\end{cases} \]

(1) \( y[-2] \) depends on \( x[-1] \) so the system has memory.

(2) Consider the input-output relations
\[
\begin{align*}
x[n] &= \delta[n] \quad \longrightarrow \quad y[n] = 0 \\
x_1[n] &= \delta[n - 1] \quad \longrightarrow \quad y_1[n] = \delta[n - 1]
\end{align*}
\]

Since \( y_1[n] \neq y[n - 1] \), the system is time varying.

(3) Assume \( x_1[n] \rightarrow y_1[n] \) and \( x_2[n] \rightarrow y_2[n] \) and \( x_3[n] = ax_1[n] + bx_2[n] \)
\[
y_3[n] = \begin{cases} 
  x_3[n], & n \geq 1 \\
  0, & n = 0 \\
  x_3[n + 1], & n \leq -1 
\end{cases} = \begin{cases} 
  ax_1[n] + bx_2[n], & n \geq 1 \\
  a \cdot 0 + b \cdot 0, & n = 0 \\
  ax_1[n + 1] + bx_2[n + 1], & n \leq -1 
\end{cases} = ay_1[n] + by_2[n]
\]

Hence the system is linear.

(4) \( y[-2] \) depends on \( x[-1] \) so the system is noncausal.

(5) If \( x[n] \leq M_x \) for all \( n \), then \( y[n] \leq M_x \) for all \( n \). Therefore, the system is BIBO stable.