A defect of the Frenet–Serret frame is the need for nondegeneracy. There is an alternative way to frame a curve without requiring nondegeneracy.

At \( s = 0 \) let \( T(0) \) denote the plane normal to \( T(0) \), and similarly \( T(s) \) denotes the plane normal to \( T(s) \). Pick a basis \( \{ M_1(0), M_2(0) \} \) for \( T(0) \) such that \( \{ T(0), M_1(0), M_2(0) \} \) constitutes a right-handed orthonormal triad. Our goal is to propagate this triad to \( \{ T(s), M_1(s), M_2(s) \} \) in such a way that certain natural condition(s) hold:

right handedness \( \Rightarrow \quad M_2(0) = T(0) \times M_1(0) \)

\[ M_2(s) = T(s) \times M_1(s) \]

Since \( T(s) - T(0) \equiv 1 \), \( T'(s) \) is \( \perp T(s) \). Hence there must exist \( k_1(s), k_2(s) \) such that

\[ T'(s) = k_1(s) M_1(s) + k_2(s) M_2(s) \]
A unit vector field \( \mathbf{M}(s) \in T(s) \) is said to be \underline{relatively parallel} along \( \gamma \) provided
\[ \mathbf{M}'(s) = f(s) \mathbf{T}(s) \]
i.e., the vector \( \mathbf{M} \) \underline{turns as little as possible}. This is the \underline{natural condition} we are looking for. We \underline{propagate} \( M_1(0), M_2(0) \) along the curve \( \gamma \) such that they remain relatively parallel at each \( s \). Thus we require
\[ M'_1(s) = f_1(s) \mathbf{T}(s) \]
\[ M'_2(s) = f_2(s) \mathbf{T}(s) \]
for some \underline{yet undetermined} \( f_i(s) \), \( i = 1, 2 \).

But
\[ M'_1(s) \cdot T(s) \equiv 0 \]
\[ \implies M'_1(s) \cdot T(s) + M_1(s) \cdot T'(s) \equiv 0 \]
\[ \iff f_1(s) + k_1(s) \equiv 0 \]
\[ \iff f_1(s) = -k_1(s) \quad i = 1, 2. \]
**Definition**  An orthonormal triad \( \{ T(s), M_1(s), M_2(s) \} \) is a relatively parallel adapted frame (RPAF) along a curve \( s \mapsto \gamma(s) \), if there exist curvature functions \( k_1(\cdot), k_2(\cdot) \), such that

\[
T'(s) = \gamma'(s)
\]

\[
T'(s) = k_1(s) M_1(s) + k_2(s) M_2(s)
\]

\[
M_1'(s) = -k_1(s) T(s)
\]

\[
M_2'(s) = -k_2(s) T(s)
\]

**Theorem**  Given a \( C^2 \) curve \( s \mapsto \gamma(s) \), and a choice \( M_1(0), M_2(0) \) in \( T(0) \perp \), such that \( \{ T(0), M_1(0), M_2(0) \} \) is a right-handed orthonormal triad, there exists a unique RPAF along \( \gamma \) that agrees with the initial choice.

**Proof:**  \[
M_1(s) = M_1(0) + \int_0^s M_1'(\tau) d\tau
\]

\[
= M_1(0) - \int_0^s k_1(\tau) T(\tau) d\tau
\]
\[ k_1(s) = \frac{d}{ds} M_1(s) \]
\[ = T'(s) \cdot M_1(s) - \int_0^s k_1(\xi) \cdot T(\xi) \cdot T(s) d\xi \]
\[ = \gamma''(s) \cdot M_1(s) - \int_0^s k_1(\xi) \gamma''(\xi) \cdot \gamma(s) d\xi \]

Similarly,
\[ k_2(s) = \gamma''(s) \cdot M_2(s) - \int_0^s k_2(\xi) \gamma''(\xi) \cdot \gamma(s) d\xi \]

These are two uncoupled Volterra integral equations of the second kind. By the standard theory of such integral equations, there exist unique \( k_i(s) \), \( i=1,2 \), solving the above equations.

There is a need to resort to numerical computation to solve the integral equations for a general curve \( \gamma \), to determine natural curvatures \( k_1, k_2 \),

* M. Bocher (1909). *An Introduction to the Study of Integral Equations; Cambridge Tracts in Mathematics and Mathematical Physics*, No. 10.
We now have a generative model for the curve \( s \mapsto \gamma(s) \) in terms of the RPAF associated to the initial frame and specified \( \text{programmed} \) curvature functions \( \kappa_i(s), \ i = 1, 2, \)

\[
\begin{bmatrix}
\frac{1}{s} M_1 & M_2 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
= \begin{bmatrix}
T & M & M_2 \\
\end{bmatrix}
\begin{bmatrix}
0 & -k_2 & 1 \\
k_1 & 0 & 0 \\
k_2 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
T(0) & M_1(0) & M_2(0) \\
\end{bmatrix}
\text{prescribed.}
\]

Relation of RPAF to Frenet-Serret

Since \( N(s), B(s) \) are in \( T_i(s) \) spanned by \( M_1(s), M_2(s) \),

\[
N(s) = \frac{1}{\kappa(s)} T'(s)
\]

\[
= \frac{1}{\kappa(s)} \left( k_1(s) M_1(s) + k_2(s) M_2(s) \right).
\]
Then

\[ 1 \equiv N(s) \cdot N(s) \]

\[ = \frac{\frac{2}{k_1(s)} + \frac{2}{k_2(s)}}{x(s)} \]

\[ \Rightarrow x(s) = \left( \frac{2}{k_1(s)} + \frac{2}{k_2(s)} \right)^{\frac{1}{2}} \]

\[ B(s) = T(s) \times N(s) \]

\[ = T(s) \times \left( \frac{k_1(s)}{k(s)} M_1(s) + \frac{k_2(s)}{k(s)} M_2(s) \right) \]

\[ = \left( -\frac{k_2(s)}{x(s)} M_1(s) + \frac{k_1(s)}{x(s)} M_2(s) \right) \]

\[ \gamma(s) = -B'(s) \times N(s) \quad \text{(torsion)} \]

\[ = -\left( -\frac{k_2}{x} M_1 + \frac{k_1}{x} M_2 \right) \left( \frac{k_1 M_1 + k_2 M_2}{x} \right) \]

\[ = \frac{1}{x^2} \left( k_2 k_1 - k_1 k_2 \right) \quad \text{(work out missing step)} \]

\[ = \left( \tan^{-1} \left( \frac{k_2}{k_1} \right) \right)' \]

\[ = \theta' \]

where \( \theta \) is polar angle in \((k_1,k_2)\) plane, ...
well-defined for \( x > 0 \).

From the integration

\[
\Theta(s) = \Theta(0) + \int_0^s \tau(r) \, dr
\]

it is clear that torsion gets accumulated in the polar angle. Since

\[
N(s) = \cos(\Theta(s)) M_1(s) + \sin(\Theta(s)) M_2(s)
\]

and

\[
B(s) = -\sin(\Theta(s)) M_1(s) + \cos(\Theta(s)) M_2(s)
\]

it is clear that \( \Theta(s) \) is the accumulated rotation (phase shift) of \( \{N(s), B(s)\} \) relative to \( \{M_1(s), M_2(s)\} \) as one proceeds along the curve from 0 to \( s \).

The plane of \( (k_1, k_2) \) is called the plane of normal development. As \( y(s) \) evolves in 3-D, the normal development \( (k_1(s), k_2(s)) \) presents a picture that captures the essential 3-D-ness of the curve \( s \mapsto y(s) \).
Exercise 4.

Suppose the curve $s \mapsto r(s)$ is confined to a plane perpendicular to vector $\mu$, not necessarily through the origin. Then the normal development is confined to a straight line passing through the origin in the $(k_1, k_2)$ plane.

Exercise 5.

Suppose the curve $s \mapsto r(s)$ is confined to a sphere of radius $R$ centered at $p \in \mathbb{R}^3$. Then the normal development is confined to a straight line in the $(k_1, k_2)$ plane at a distance $\frac{1}{R}$ from $(0, 0)$.

Remark One should think of the normal development of $s \mapsto r(s)$ as a signature curve.