Examples of Smooth Manifolds

1. Unit sphere \( S^2 = \{ (x, y, z) : x^2 + y^2 + z^2 = 1 \} \)

\[ W_1 = \{ (x, y) : x^2 + y^2 < 1 \} \subset \mathbb{R}^2 \]

\[ \eta : W_1 \rightarrow S^2 \]

\( (x, y) \rightarrow (x, y, \sqrt{1-x^2-y^2}) \)

is a diffeomorphism of \( W_1 \) onto the region \( z > 0 \) of \( S^2 \) — i.e., a local parametrization.

By interchanging the roles of \( x, y, z \) and changing the signs of the variables, we obtain parametrizations of the regions \( x > 0, y > 0, z < 0, y < 0 \) and \( z < 0 \). Since these cover \( S^2 \), it follows that \( S^2 \) is a smooth manifold. \( \square \) [We have checked condition 1]

[What about condition 2?]

2. \( \mathbb{R}^k \) is itself a differentiable manifold.

\( W = \mathbb{R}^k \)

\( \eta = \text{identity map} \).

3. Let \( M^k \) be a smooth manifold.

Let \( U \) be an open subset of \( M^k \).

Then \( U \) is also a manifold, an open submanifold of dimension \( k \).
Proof: $p \times p$ Matrices $A$ determinant $\neq 0$ open $C$ all $p \times p$ matrices $\in \mathbb{R}^F$