Homework Set #1 (to be returned February 8)

1. Suppose $X$ and $Y$ are smooth manifolds. Show that
   (a) $X \times Y$ is a smooth manifold
   (b) $\dim(X \times Y) = \dim(X) + \dim(Y)$
   (c) $\pi_1 : X \times Y \to X$ defined by $\pi_1(x, y) = x$ is a smooth map.

2. Exhibit enough parametrizations to cover $S^1 \times S^1 \subset \mathbb{R}^4$.

3. Suppose $ad - bc > 0$. Show that there is just one way to express the matrix $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$ as a product
   $$(\begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{smallmatrix}) \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$
   with $p > 0, r > 0$. Deduce that the space $\{ (a, b, c, d) \in \mathbb{R}^4 : ad - bc > 0 \}$ is homeomorphic to $S^1 \times \mathbb{R}^3$. Is it diffeomorphic?

4. Suppose $M \subset \mathbb{R}^n$ is given the subspace topology and is viewed as a smooth manifold in the sense that, each $x \in M$ has a neighborhood $U_x \subset M$ that is diffeomorphic to an open set $W_x \subset \mathbb{R}^k$ with diffeomorphism $\phi_x : U_x \ni M \to W_x$. Further suppose that $(U_x', \phi_x)$ and $(U_y', \phi_y)$ are overlapping coordinate charts, i.e., $U_x' \cap U_y' \neq \emptyset$. Then show that
   $\phi_x \circ \phi_y^{-1} : \phi_y(U_x' \cap U_y') \to \mathbb{R}^k$ is smooth.