We derive some properties of gradient algorithms (and slight variants)

First some notation; For $1 \le p < \infty$,

$$L_p^m = \left\{ f : [0, \infty) \to \mathbb{R}^m \;\middle|\; \int_0^\infty \| f(t) \|^p \, dt < \infty \right\}.$$

Here $\| \ \|$ denotes any norm on the finite dimensional space $\mathbb{R}^m$. All such norms being equivalent,* if the inequality in the definition above holds in one norm on $\mathbb{R}^m$, it is true in any other norm on $\mathbb{R}^m$.

If $f \in L_p^m$ then we define the <u>function space norm</u>

$$\| f \|_p = \left(\int_0^\infty \| f(t) \|^p \right)^{\frac{1}{p}}.$$

$$L_\infty^m = \left\{ f : [0, \infty) \to \mathbb{R}^m \;\middle|\; \forall t \ge 0,\ \| f(t) \| < M \text{ for some } M > 0 \right\}$$

We then define the <u>function space</u> norm

$$\| f \|_\infty = \sup_{t \ge 0} \| f(t) \|$$

on the function space L_∞^m.

It is convenient to drop the superscript m in L_∞^m, L_p^m etc. as it will be apparent

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* Two norms $\| \ \|^a$ and $\| \ \|^b$ are equivalent if there exist constants c_1 and c_2 both > 0 s.t. $c_1 \| x \|^a \le \| x \|^b \le c_2 \| x \|^a \quad \forall x$.

from the context where the functions take values.

Gradient algorithm properties

(a) Consider $\dot{\phi} = \dot{\theta} = -\gamma e_1 w$

$$e_1 = \phi^T w \qquad \gamma > 0$$

where $w : \mathbb{R}_+ \to \mathbb{R}^{2n}$ is assumed to be piecewise continuous. Then,

$$e_1 \in L_2 \quad \text{and} \quad \phi \in L_\infty$$

Proof
$$\dot{\phi} = -\gamma w w^T \phi$$

$V(\phi) = \frac{1}{2} \phi^T \phi$ satisfies

$$\dot{V} = \phi^T \dot{\phi} = -\gamma \phi^T w w^T \phi$$
$$= -\gamma (w^T \phi)^2$$
$$\leq 0$$

Hence,

$$0 \leq V(\phi(t)) \leq V(\phi(0)) \quad \forall t \geq 0 \Rightarrow \phi \in L_\infty .$$

$V(\phi(t))$ is monotone decreasing and bounded below.

Hence $\lim_{t \to \infty} V(\phi(t))$ exists and is finite $= V_\infty$

But
$$\int_0^\infty e^2(t)\,dt = \int_0^\infty (\phi^T(t)\, w(t))^2\,dt$$
$$= \int_0^\infty - \frac{\dot{V}(\phi(t))}{\gamma}\,dt$$

$$= \frac{V(\phi(0)) - V_\infty}{\gamma} < \infty .$$

Thus $\quad e \in L_2$ ▨

(b) Consider $\quad \dot\phi = \dot\theta = -\dfrac{\gamma\, e_1 w}{1 + \varepsilon_0 w^T w} \qquad \varepsilon_0 > 0 , \ \gamma > 0$

and $e_1 = \phi^T w$,
(we have a normalized gradient). Assome

$w : \mathbb{R}_+ \to \mathbb{R}^{2n}$ is piecewise continuous and $e_1 = \phi^T w$.

Then (i) $\quad \dfrac{e_1}{\sqrt{1 + \varepsilon_0 w^T w}} \in L_2 \cap L_\infty$

(ii) $\quad \phi \in L_\infty , \quad \dot\phi \in L_2 \cap L_\infty$

(iii) $\quad \beta = \dfrac{\phi^T w}{1 + \| w_t \|_\infty} \in L_2 \cap L_\infty$

(here $\quad \| w_t \|_\infty = \max\limits_{i = 1, 2, \dots, 2n} |w_i(t)|$)

Proof. Let $\quad V = \phi^T \phi$

Then $\dot V \cancel{\phi} = 2 \phi^T \dot\phi = 2 \phi^T \left(\dfrac{-\gamma\, e_1 w}{1 + \varepsilon_0 w^T w} \right)$

$$= - \frac{2\gamma\, \phi^T w w^T \phi}{1 + \varepsilon_0 w^T w} = \frac{-2\gamma\, e_1^2}{1 + \varepsilon_0 w^T w} \le 0$$

Then $\quad 0 \le V(\phi(t)) \le V(\phi(0)), \quad \forall\, t \geqslant 0 .$

$V \in L_\infty$

ϕ satisfies $\quad \|\phi(t)\|_2 = \sqrt{\sum_{i=1}^{2n} (\phi_i(t))^2} \leq \sqrt{V(\phi(0))} \quad \forall t \geq 0$

$$\Rightarrow \quad \phi \in L_\infty$$

$\dfrac{e_1}{\sqrt{1+\varepsilon_0 w^T w}}$ satisfies, $\quad \left| \dfrac{e_1(t)}{\sqrt{1+\varepsilon_0 w^T(t) w(t)}} \right| = \left| \dfrac{\phi(t)^T w(t)}{\sqrt{1+\varepsilon_0 w^T(t) w(t)}} \right|$

$$= \frac{|\phi^T(t) \; w(t)|}{\sqrt{1+\varepsilon_0 \; w^T(t) \, w(t)}}$$

$$\leq \frac{\|\phi(t)\|_2 \; \|w(t)\|_2}{\sqrt{1+\varepsilon_0 \; w^T(t) \, w(t)}} \qquad \text{(Cauchy-Schwarz)}$$

$$\leq \sqrt{V(\phi(0))} \cdot \frac{1}{\sqrt{\varepsilon_0}} \sqrt{\frac{w^T(t) \, w(t)}{\frac{1}{\varepsilon_0} + w^T(t) \, w(t)}}$$

$$\leq \frac{1}{\sqrt{\varepsilon_0}} \sqrt{V(\phi(0))}$$

$$\Rightarrow \quad \frac{e_1}{\sqrt{1+\varepsilon_0 w^T w}} \in L_\infty$$

$\beta = \dfrac{\phi^T w}{1 + \|w_t\|_\infty}$ satisfies

$$|\beta(t)| = \frac{|\phi^T(t) \; w(t)|}{1 + \max_i |w_i(t)|} \leq \sum_{j=1}^{2n} |\phi_j(t)| \frac{|w_j(t)|}{1 + \max_i |w_i(t)|}$$

$$\leq \sum_{j=1}^{2n} |\phi_j(t)|$$

$$\leq \; 2n \; \|\phi(t)\|_2 \; \leq \; 2n\sqrt{V(\phi(0))}$$

$\Rightarrow \; \beta \in L_\infty$.

We have shown, V, ϕ, $\dfrac{e_1}{\sqrt{1+\varepsilon_0 w^T w}}$, β all belong to L_∞.

Now $$\dot{\phi} = \frac{-\gamma \, w \, e_1}{1+\varepsilon_0 \, w^T w}$$

$$= -\frac{\gamma}{\varepsilon_0} \, \varepsilon_0 \frac{w \, w^T \phi}{1+\varepsilon_0 w^T w}$$

$$\|\dot{\phi}(t)\|_2 = \frac{\gamma}{\varepsilon_0} \left\| \varepsilon_0 \frac{w(t) \, w^T(t) \, \phi(t)}{1+ \varepsilon_0 \, w^T(t) \, w(t)} \right\|_2$$

$$= \frac{\gamma}{\varepsilon_0} \cdot \frac{1}{(1+\varepsilon_0 \, w^T(t) \, w(t))} \cdot \varepsilon_0 \; \| w(t) \, w^T(t) \, \phi(t) \|_2$$

Recall that $$\|Ax\|_2 = \sqrt{x^T A^T A x}$$

$$\leq \sqrt{\lambda_{max}(A^T A)} \; \sqrt{x^T x}$$

$$= \sqrt{\lambda_{max}(A^T A)} \; \|x\|_2 .$$

For $A = w(t) w^T(t)$ $\quad \lambda_{max}(A^T A)$

$$= \lambda_{max} \left(w(t) \, w^T(t) \, w(t) \, w^T(t) \right)$$

$$= \lambda_{max} \left((w^T(t) \, w(t)) \quad w(t) \, w^T(t) \right)$$

$$= \left(w^T(t) \, w(t) \right)^2 .$$

Hence $\| w(t) w^T(t) \phi(t) \|_2$

$$\leq \sqrt{\lambda_{max}\left(\left(w(t) w^T(t)\right)^T \left(w(t) w^T(t)\right) \right)} \quad \| \phi(t) \|_2$$

$$= \sqrt{\left(w^T(t) w(t)\right)^2} \quad \| \phi(t) \|_2$$

$$= w^T(t)\, w(t) \quad \| \phi(t) \|_2 \, .$$

Hence

$$\| \dot{\phi}(t) \|_2 \leq \frac{\gamma}{\varepsilon_0} \frac{1}{\left(1 + \varepsilon_0 w^T(t) w(t)\right)} \left(\varepsilon_0 \; w^T(t) w(t) \right) \quad \| \phi(t) \|_2$$

$$\leq \frac{\gamma}{\varepsilon_0} \| \phi(t) \|_2$$

$$\leq \frac{\gamma}{\varepsilon_0} \sqrt{V(\phi(0))}$$

$$\Rightarrow \dot{\phi} \in L_\infty .$$

$$\int_0^\infty \left[\frac{e_1(t)}{\sqrt{1 + \varepsilon_0 w^T(t) w(t)}} \right]^2 dt = \int_0^\infty \frac{e_1^2(t)}{1 + \varepsilon_0 w^T(t) w(t)} dt$$

$$= \int_0^\infty - \frac{\dot{V}(\phi(t))}{2\gamma} dt$$

$$= \frac{V(\phi(0)) - V_\infty}{2\gamma}$$

exists and is finite since $V(\phi(t))$ is monotone decreasing & bounded below.

Thus $\dfrac{e_1}{\sqrt{1+\varepsilon_0 W^T W}} \in L_2$

We have thus far shown $\dfrac{e_1}{\sqrt{1+\varepsilon_0 W^T W}} \in L_2 \cap L_\infty$.

(completes (i))

$$\beta(t) = \frac{\phi^T(t)\, W(t)}{1 + \max\limits_{1\le i\le 2n} |W_i(t)|}$$

$$= \frac{\phi^T(t)\, W(t)}{\sqrt{1+\varepsilon_0 W^T(t)\, W(t)}} \quad \frac{\sqrt{1+\varepsilon_0 W^T(t)\, W(t)}}{1 + \max\limits_{1\le i\le 2n} |W_i(t)|}$$

This belongs to L_2 (see above)

if we show this belongs to L_∞ we are done.

Recall, in finite dimensions.

$$\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\, \|x\|_\infty .$$

Hence
$$\frac{\sqrt{1+\varepsilon_0 \|W(t)\|_2^2}}{1 + \max\limits_{1\le i\le 2n} |W_i(t)|} \le \frac{\sqrt{1+\varepsilon_0 \|W(t)\|_2^2}}{1 + \frac{1}{\sqrt{n}} \|W(t)\|_2}$$

$$f(y) = \frac{1 + \varepsilon_0 y^2}{\left(1 + \frac{1}{\sqrt{n}} y\right)^2} = \frac{1 + \varepsilon_0 y^2}{1 + \frac{1}{n} y^2 + \frac{2}{\sqrt{2}} y}$$

is continuous, positive on $y > 0$ and $\to n\varepsilon_0$ as $y \to \infty$. Pick a $\delta > 0$. Then $\exists\, Y > 0$ such that $\forall\, y > Y \Rightarrow f(y) < n\varepsilon_0 + \delta$.

On the other hand, on the compact set $[0, Y]$ $f(y)$ has a maximum f_{max} due to continuity of $f(y)$. (Weierstrass's Theorem)

Hence $f(y) < \max(f_{max}, n\varepsilon_0 + \delta)$

$\forall\, y \geq 0$.

$\Rightarrow \sqrt{f} \in L_\infty$.

Thus we have shown

$\beta \in L_2$.

Hence $\beta \in L_2 \cap L_\infty$ (complete (iii))

$$\|\dot{\phi}(t)\|_2^2 = \frac{\gamma^2 e_1^2(t)\, W^T(t) W(t)}{(1 + \varepsilon_0 W^T(t) W(t))^2}$$

$$= \frac{\gamma^2}{\varepsilon_0} \frac{e_1^2(t)}{(1 + \varepsilon_0 W^T(t) W(t))} \frac{\varepsilon_0 W^T(t) W(t)}{(1 + \varepsilon_0 W^T(t) W(t))}$$

$$\leq \frac{\gamma^2}{\varepsilon_0} \frac{e_1^2(t)}{(1 + \varepsilon_0 W^T(t) W(t))}$$

We have already shown

$$\frac{e_1}{\sqrt{1+\varepsilon_0 w^T w}} \in L_2$$

Hence $\quad \dot{\phi} \in L_2$.

Hence $\quad \cancel{\phi}\ \dot{\phi} \in L_2 \cap L_\infty \qquad$ (completes (ii))

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c) Suppose $\quad \dot{\phi} = \dot{\theta} = -\gamma\, e_1 w \qquad \gamma > 0$

$$e_1 = \phi^T w + \varepsilon$$

where $\varepsilon(t) \to 0$ exponentially in t.

Suppose $w : \mathbb{R}_+ \to \mathbb{R}^{2n}$ is piecewise continuous.

Then $\quad e_1 \in L_2 \quad , \quad \phi \in L_\infty$.

Proof: Define $\quad V \triangleq \phi^T \phi + \frac{\gamma}{2} \int_t^\infty \varepsilon^2(\sigma)\, d\sigma$

bounded by hypothesis on $\varepsilon(t)$

Then $\quad \dot{V} = 2\phi^T \dot{\phi} - \frac{\gamma}{2}\varepsilon^2(t)$

$$= 2\phi^T\left(-\gamma\,(\phi^T w + \varepsilon)\, w\right) - \frac{\gamma}{2}\varepsilon^2$$

$$= -2\gamma\,(\phi^T w)^2 - 2\gamma(\phi^T w)\,\varepsilon - \frac{\gamma}{2}\varepsilon^2$$

$$= -2\gamma\left(\phi^T w + \frac{\varepsilon}{2}\right)^2 \le 0$$

$0 \le V$ and V is monotone decreasing with t.

$$\Rightarrow \quad \lim_{t \to \infty} V(t) \quad \text{exists and is finite} = V_\infty$$

$$\|\phi(t)\|_2 \le \sqrt{V(0)} = \left(\|\phi(0)\|_2^2 + \frac{\gamma}{2} \int_0^\infty \varepsilon^2(\sigma)\, d\sigma \right)^{1/2}$$

$$\Rightarrow \quad \phi \in L_\infty$$

$$\int_0^\infty \left(\phi^T(t)\, w(t) + \frac{\varepsilon(t)}{2} \right)^2 dt$$

$$= \int_0^\infty - \frac{\dot{V}}{2\gamma}\, dt$$

$$= \frac{V(0) - V_\infty}{2\gamma} < \infty$$

Thus $\quad \phi^T w + \frac{\varepsilon}{2} \in L_2$

On ~~the~~ other hand $\quad \frac{\varepsilon}{2} \in L_2$

Hence $\quad e_1 = \left(\phi^T w + \frac{\varepsilon}{2} \right) + \frac{\varepsilon}{2} \in L_2$

(recall L_2 is a vector space). ▨

Least Squares Algorithm (with normalization and Covariance resetting)

$$\dot{\phi} = \dot{\theta} = -\frac{\gamma P w e_1}{1 + \varepsilon_0 w^T w} \qquad \gamma > 0 \quad \varepsilon_0 > 0$$

and

$$\dot{P} = \frac{-\gamma P w w^T P}{1 + \varepsilon_0 w^T P w}$$

$$P(t_0) = P(t_r^+) = k_0 \mathbb{1} > 0 \qquad \text{(resetting)}$$

where

$$t_r = \{ t \mid \lambda_{min}(P(t)) \leq k_1 < k_0 \}.$$

Suppose $w : \mathbb{R}_+ \to \mathbb{R}^{2n}$ is piecewise continuous. Then

(i) $\dfrac{e_1}{\sqrt{1 + \varepsilon_0 w^T P w}} \in L_2 \cap L_\infty$

(ii) $\phi \in L_\infty$, $\quad \dot{\phi} \in L_2 \cap L_\infty$

(iii) $\beta = \dfrac{\phi^T w}{1 + \| w_t \|_\infty} \in L_2 \cap L_\infty$

PROOF : Homework Exercise.