We let $(\Omega, \mathcal{A}, P)$ denote a probability space with $\Omega$: set of elementary events, $\mathcal{A}$: Sigma algebra of events (subsets of $\Omega$) and $P: \mathcal{A} \rightarrow [0, 1]$ a probability measure.

A random variable $X: \Omega \rightarrow \mathbb{R}$ is such that $X^{-1}(B) \in \mathcal{A}$ whenever $B \in \mathcal{B} = \text{Borel Sigma algebra of } \mathbb{R}$. (measurable function on $\Omega$).

A stochastic process-indexed by $T$ is

$$\{ X_t : t \in T \}$$

with each $X_t: \Omega \rightarrow \mathbb{R}$ a random variable.

Distribution function(s)

$$F_X(x) = P \{ \omega : X(\omega) \leq x \}$$

$$F_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) = P \{ \omega : X_i(\omega) \leq x_i, \quad i = 1, 2, \ldots, n \}$$

$n$ a positive integer

We will often write

$x(t)$ or $X(t)$ when we mean $X_t(t)$

$$E(X) = \int x \, dF_X(x)$$

denote the expectation of $X$
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1. Kalman-Bucy Filtering and Duality

We are interested in estimating (reconstructing) the state of a linear system driven by noise, using linear measurements of the state corrupted by noise. Under suitable assumptions about the noise processes and a measure of quality of state estimate, we derive a dynamical system that propagates an optimal state estimate. This system is linear, and is driven by the measurements. In discrete time, it is called the Kalman filter, and in continuous time it is due to Kalman and Bucy. The derivation is based on solving a dual, deterministic optimal control problem.

Consider the stochastic system

\[ \begin{align*}
    dx &= Ax dt + dW \\
    dy &= Cx dt + d\xi
\end{align*} \]  

(1)

Here \( \{ W(t) : t \in T \} \) and \( \{ \xi(t) : t \in T \} \) are orthogonal increment processes with parameters \( R \), and \( R_2 \), respectively, and are mutually orthogonal as well.
Specifically, \( E(v(t)) = 0 \) and \( E(e(t)) = 0 \).

Given \( t_1 < t_2 < t_3 < t_4 \),

\[
\begin{align*}
(v(t_2) - v(t_1)) & \perp (v(t_4) - v(t_3)) \\
(e(t_2) - e(t_1)) & \perp (e(t_4) - e(t_3)) \\
v(t) & \perp e(s)
\end{align*}
\]

\[
E \left( (v(t_2) - v(t_1))(v(t_2) - v(t_1))^T \right) = \int_{t_1}^{t_2} R(s) \, ds
\]

\[
E \left( (e(t_2) - e(t_1))(e(t_2) - e(t_1))^T \right) = \int_{t_1}^{t_2} R(s) \, ds
\]

Here \( R_1(s) = R_1(s) \geq 0 \) and \( R_2(s) = R_2(s) \geq 0 \).

Understanding equation (1) is one of our goals.

Integrating both sides of (1), we get the integral eqn.

\[
X(t) = X(t_0) + \int_{t_0}^{t} A(s) X(s) \, ds + \int_{t_0}^{t} dV(s) = X(t_0) + \int_{t_0}^{t} A(s) X(s) \, ds + V(t) - V(t_0) \tag{2}
\]

and,

\[
Y(t) = Y(t_0) + \int_{t_0}^{t} C(s) X(s) \, ds + e(t) - e(t_0)
\]

Here we have assumed that the integrals, properly defined, satisfy the ordinary rules of calculus.
The simply interprets (1) to mean (2). Formally, the processes defined by

\[ x(t) = \Phi(t, t_0) x(t_0) + \int_{t_0}^{t} \Phi(t, \tau) d\nu(\tau) \]

\[ y(t) = y(t_0) + \int_{t_0}^{t} c(\tau) x(\tau) d\sigma + e(t) - e(t_0) \]

solve (2). Making sense of the integrals, here is important. We will do this in lecture (2).

First, let \( E(x(t_0)) = m_0 \)

**Problem of state estimation.**

Determine a process \( \hat{x}(t) \) such that, for any constant \( a \) and a suitable constant \( b \),

\[(3) \quad a^T \hat{x}(t_1) = -\int_{t_0}^{t_1} a^T y(\tau) d\nu(\tau) + b^T m_0 \]

is the minimum mean square error (MMSE) estimate of \( a^T x(t) \).

The parameters to be decided are \( u(\cdot) \) and \( b \), to minimize \( E((a^T x(t_1) - a^T \hat{x}(t_1))^2) = \text{cost} \)
We calculate

\[
\mathbf{a}^T \mathbf{x}(t) = -\int_{t_0}^{t_1} \mathbf{u}(\sigma) \mathrm{d}y(\sigma) + b \mathbf{m}_0
\]

\[
= -\int_{t_0}^{t_1} \mathbf{u}(\sigma) \mathbf{c}(\sigma) \mathbf{x}(\sigma) \mathrm{d}\sigma - \int_{t_0}^{t_1} \mathbf{u}(\sigma) \mathrm{d}e(\sigma) + b \mathbf{m}_0
\]

For any differentiable deterministic function \( z(\cdot) \), with \( z(t_1) = \mathbf{a} \),

\[
\mathbf{a}^T \mathbf{x}(t_1) = z^T(t_1) \mathbf{x}(t_1)
\]

\[
= z^T(t_0) \mathbf{x}(t_0) + \int_{t_0}^{t_1} z^T(\sigma) \mathbf{x}(\sigma) \mathrm{d}\sigma
\]

\[
= z^T(t_0) \mathbf{x}(t_0) + \int_{t_0}^{t_1} z^T(\sigma) \mathbf{d}x(\sigma) \quad \text{(fundamental theorem of integral calculus)}
\]

\[
= z^T(t_0) \mathbf{x}(t_0) + \int_{t_0}^{t_1} z^T(\sigma) \mathbf{x}(\sigma) \mathrm{d}\sigma + \int_{t_0}^{t_1} z^T(\sigma) \mathbf{d}x(\sigma)
\]

Suppose \( \mathbf{x} \) is defined by the dual system

\[
\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = -\mathbf{A}^T \mathbf{z} - \mathbf{C}^T \mathbf{u} \quad \text{with} \quad \mathbf{z}(t_1) = \mathbf{a}.
\]

Then

\[
\mathbf{a}^T \mathbf{x}(t) = z^T(t_0) \mathbf{x}(t_0) + \int_{t_0}^{t_1} (-z^T(\sigma) \mathbf{A}(\sigma) \mathbf{x}(\sigma) - z^T(\sigma) \mathbf{c}(\sigma)) \mathrm{d}\sigma
\]

\[
+ \int_{t_0}^{t_1} z^T(\sigma) \mathbf{A}(\sigma) \mathbf{x}(\sigma) \mathrm{d}\sigma + z^T(\sigma) \mathbf{d}x(\sigma)
\]
\[ = z_T(t_0) x(t_0) + \int_{t_0}^t \left( -u^T(\tau) c(\tau) x(\tau) d\tau + \hat{z}(\tau) d\nu(\tau) \right) \]

Sum (5) and (6), we get,

\[ a^T(t_1) - \hat{x}(t_1) = \left( z^T(t_0) x(t_0) + b^T m_0 \right) \]

\[ + \int_{t_0}^{t_1} \left( z^T(\tau) d\nu(\tau) + u^T(\tau) d\varepsilon(\tau) \right) \]

As yet \( x \) and \( u \) are undetermined but deterministic.

Taking expectation, we get

\[ E \left( a^T(t_1) - \hat{x}(t_1) \right) \]

\[ = \left( z^T(t_0) x(t_0) - b^T m_0 \right) \]

which should be necessarily 0 for a MMSE estimate.

\[ \Rightarrow \text{Set } b = z(t_0) \]

(Which in turn is to be determined from \( u \).

Then performance objective for filter is to minimize
\[ E (a^T x(t_i) - a^T \bar{x}(t_i))^2 \]

\[ = \text{Var} \left( Z(t_0) (x(t_0) - m_o) + \int_{t_0}^{t_i} \overline{z(t_0)} d\nu(t) + u(t_0) d\omega(t) \right) \]

\[ = Z(t_0) R_0 Z(t_0) + \int_{t_0}^{t_i} \overline{Z(t_0)} R_1(t_0) \overline{Z(t_0)} dt \]
\[ + \int_{t_0}^{t_i} \overline{u(t_0)} R_2(t_0) u(t_0) dt \]

--- (7) ----

Here we have used \( X_0 \perp V(t), \ e(t), \ t_s > 0 \) and\[ \mathbb{E} (x(t_0) - m_o)^T (x(t_0) - m_o)^T = R_0. \]

Thus the MMSE estimation problem becomes a (backward* time) optimal control problem

\[ \dot{z} = -A^T z - C^T u \]
\[ z(t_i) = 0 \]

Minimize \[ \gamma \]

\[ u(t) \]

\[ \gamma = Z(t_0) R_0 Z(t_0) + \int_{t_0}^{t_i} \overline{Z(t_0)} R_1(t_0) \overline{Z(t_0)} + \overline{u(t_0)} R_2(t_0) u(t_0) dt \]
By the fundamental lemma (on path independent integrals) (see ENEE 664 Lecture 2)

\[
\begin{align*}
Z(t_1) P(t_1) Z(t_1) - Z(t_0) P(t_0) Z(t_0) &= \int_{t_0}^{t_1} \left( u^T(\sigma) \dot{Z}(\sigma) \right) \\
&= \int_{t_0}^{t_1} \left( u^T(\sigma) \dot{Z}(\sigma) \right) \\
&= \int_{t_0}^{t_1} \left( \begin{array}{c}
0 \\
\dot{P}(t) Z(t) \\
\dot{P}(t) A(t) P(t) \\
-P(t) A(t) \\
P(t) A(t)
\end{array} \right) Z(t) \\
&= \int_{t_0}^{t_1} \left( \begin{array}{c}
0 \\
\dot{P}(t) Z(t) \\
\dot{P}(t) A(t) P(t) \\
-P(t) A(t) \\
P(t) A(t)
\end{array} \right) Z(t)
\end{align*}
\]

for \( Z, u \) satisfying the above model, and any differentiable matrix function \( P(t) = P(t) \).

Then the objective function \( \mathcal{J} \)

\[
\begin{align*}
\mathcal{J} &= Z(t_0) R_0 Z(t_0) + \int_{t_0}^{t_1} \left( Z(\sigma) R_1(\sigma) Z(\sigma) + u^T(\sigma) R_2(\sigma) u(\sigma) \right) d\sigma \\
&= Z(t_0) R_0 Z(t_0) + \int_{t_0}^{t_1} \left( Z(\sigma) R_1(\sigma) Z(\sigma) \right) \\
&= \int_{t_0}^{t_1} \left( u^T(\sigma) \dot{Z}(\sigma) \right) \\
&= \int_{t_0}^{t_1} \left( u^T(\sigma) \dot{Z}(\sigma) \right) \\
&= \int_{t_0}^{t_1} \left( \begin{array}{c}
0 \\
\dot{P}(t) Z(t) \\
\dot{P}(t) A(t) P(t) \\
-P(t) A(t) \\
P(t) A(t)
\end{array} \right) Z(t)
\end{align*}
\]

for \( Z, u \) satisfying the above model, and any differentiable matrix function \( P(t) = P(t) \).
Suppose \( P = P^T \) satisfies the Riccati equation
\[
\dot{P} = AP + PA^T + R_1 - PC^T R_2^{-1} CP
\]
\( P(t_0) = R_0 \)
on the entire interval \([t_0, t_1]\).

Then
\[
Y = Z(t_1) P(t_1) Z(t_1)^T \\
+ \int_{t_0}^{t_1} \begin{pmatrix} R_2(\tau) & (\phi(\tau) P(\tau)) \\ (\phi(\tau)^T P(\tau) C(\tau)) & P(\tau) (\phi(\tau) R_2^{-1}(\tau) C(\tau) P(\tau))^T \end{pmatrix} Z(\tau) \, d\tau
\]

\[
= Z(t_1) P(t_1) Z(t_1)^T \\
+ \int_{t_0}^{t_1} \begin{pmatrix} R_2(\tau) & (\phi(\tau) P(\tau)) \\ (\phi(\tau)^T P(\tau) C(\tau) R_2^{-1}(\tau) C(\tau) P(\tau))^T \end{pmatrix} Z(\tau) \, d\tau
\]

which is minimized when
\[
u(\sigma) = - R_2^{-1}(\sigma) P(\sigma) Z(\sigma)
\]
to \(0 \leq \sigma \leq t_1\).
Summarizing, the MMSE estimate of $\hat{x}(t)$ is given by
$$ a^T \hat{x}(t) = -\int_t^T u(\tau) d\gamma(\tau) + b^T w_0 $$

where
$$ u(t) = -R_2^{-1}(t) C(t) P(t) \hat{x}(t) $$
$$ \dot{x}(t) = -A^T z(t) - C^T u(t) $$
$$ P = AP + PA^T + R_1 - P C R_2^{-1} P $$
$$ P(t_0) = R_0, \quad i \neq t_1 = a $$
$$ b = \hat{x}(t_0) $$

We can turn the estimates above into a recursive filter very easily.

Let $K = PC(t) C(t) R_2^{-1}(t)$
$$ \dot{z} = -A^T z + C^T K \dot{z} = -A z + \dot{K} z $$
$$ -9- $$
let \( \Phi(t, t_1) \) be the solution to the matrix equation
\[
\frac{d\Phi}{dt} = (A - K(t)) \Phi
\]
\( \Phi(t_1, t_1) = 1 \) the identity matrix.

Then
\[
\dot{z}(t) = \Phi^T(t_1, t) \dot{z}(t_1)
\]
(recall adjoints)

\[
= -\Phi^T(t_1, t) \alpha.
\]

Then
\[
\dot{u}(t) = -K(t) \Phi^T(t_1, t) \alpha
\]
\[
b = z(t_0) = \Phi^T(t_1, t_0) \alpha.
\]

Then
\[
\alpha^T \dot{x}(t_1) = -\int_{t_0}^{t_1} \Phi^T(t_1, \tau) K(t) \Phi(t_1, \tau) d\tau + b^T \mu_0
\]
\[
= -\int_{t_0}^{t_1} \Phi^T(t_1, \tau) K(t) \Phi(t_1, \tau) d\tau
\]
\[
= \alpha^T \int_{t_0}^{t_1} \Phi(t_1, t) K(t) \Phi(t_1, t) d\tau + \alpha^T \Phi(t_1, t_0) \mu_0.
\]

This equation is satisfied if we define
\[
\dot{x}(t_1) = \int_{t_0}^{t_1} \Phi(t_1, t) K(t) \Phi(t_1, t) d\tau + \Phi(t_1, t_0) \mu_0
\]
to be the state estimate. We recursively it by
\[
\dot{x}(t_1) = \Phi(t_1, t_1) K(t_1) \Phi(t_1, t_1) d\dot{y}(t_1)
\]
\[
+ \int_{t_0}^{t_1} \frac{\partial \Phi(t_1, t)}{\partial t} K(t) \Phi(t_1, t) d\tau + \frac{\partial \Phi(t_1, t_0)}{\partial t} \mu_0 dt.
\]
\[ = K(t_i) dy(t_i) + \left( \int_{t_0}^{t_i} (A(t_i) - K(t_i) C(t_i)) \tilde{F} (t, t) K(t_i) dy(t) \right) dt_i \\
+ (A(t_i) - K(t_i) C(t_i)) \tilde{F} (t_i, t_0) m_0 dt_i \]

\[ = K(t_i) dy(t_i) + (A(t_i) - K(t_i) C(t_i)) \hat{x}(t_i) dt_i. \]

\[ A(t_i) \hat{x}(t_i) dt_i + K(t_i) (dy(t_i) - C(t_i) \hat{x}(t_i) dt_i) \]

The term multiplying \( K(t_i) \) is referred to as the **innovation** (i.e. new information supplied by the incremental observation \( dy \)).

We have derived the **Kalman-Bucy filter** (the continuous-time analog of the Kalman filter in discrete time) \( \to \) see app. E. Bertsekas vol 1

**Theorem (K-B):** The optimal (MMSE) estimate is computed recursively by

\[ \dot{x} = A \hat{x} \text{ } dt + K (dy - C \hat{x} \text{ } dt) \]

\[ \hat{x}(t_0) = m_0 \]

\[ K = PC^T R_2^{-1} \]

\[ \dot{P} = AP + PA^T + R_1 - PC^T R_2^{-1} C P \]

\[ P(t_0) = R_0 \]
Solving \( dx = A x dt + dv \) \hspace{1cm} (1)

Observe that (1) is equivalent to

\[
x(t) = x(t_0) + \int_{t_0}^{t} A(\sigma) x(\sigma) \, d\sigma + v(t) - v(t_0)
\]

Let \( z(t) = \Phi(t, t_0)^{-1} x(t) \), then, \( z(t) \) from (2),

\[
z(t) = \Phi(t, t_0)^{-1} \, z(t_0)
\]

\[
+ \Phi(t, t_0)^{-1} \left\{ \int_{t_0}^{t} \Phi(\sigma, t_0) \, A(\sigma) \, z(\sigma) \, d\sigma \right\}
\]

\[
+ \Phi(t, t_0)^{-1} \left( v(t) - v(t_0) \right)
\]

\[
= \Phi(t_0, t) \, z(t_0)
\]

\[
+ \Phi(t_0, t) \left\{ \int_{t_0}^{t} \Phi(\sigma, t_0) \, A(\sigma) \, z(\sigma) \, d\sigma \right\}
\]

\[
+ \Phi(t_0, t) \left( v(t) - v(t_0) \right)
\]

\[ \text{(Integration by parts)} \]

\[
= \Phi(t_0, t) \, z(t_0) + z(t) - \Phi(t_0, t) \, z(t_0)
\]

\[
- \Phi(t_0, t) \left[ \int_{t_0}^{t} \Phi(\sigma, t_0) \, d\sigma \right]
\]

\[
+ \Phi(t_0, t) \left( v(t) - v(t_0) \right)
\]

\[ \leq \int_{t_0}^{t} \Phi(\sigma, t_0) \, d\sigma \leq v(t) - v(t_0) \quad \cdots \quad (4)
\]
\[
\Phi(t, t_0) \, dZ(t) = dV(t) \quad -- (5)
\]

\[
\Rightarrow \quad dZ(t) = \Phi(t_0, t) \, dV(t)
\]

\[
\Rightarrow \quad Z(t) = Z(t_0) + \int_{t_0}^{t} \Phi(t_0, \sigma) \, dV(\sigma) \quad -- (6)
\]

\[
\Rightarrow \quad X(t) = \Phi(t, t_0) \, Z(t)
\]

\[
= \Phi(t, t_0) \, Z(t_0) + \int_{t_0}^{t} \Phi(t_0, \sigma) \, dV(\sigma)
\]

\[
= \Phi(t, t_0) \, X(t_0) + \int_{t_0}^{t} \Phi(t_0, \sigma) \, dV(\sigma) \quad -- (7)
\]

the stochastic variation of constants form