Lecture 1 (b) Fall 2010 ENEE 660

10. We have made a distinction between linear maps and matrices that represent them. Representations require choice of bases. The matrix elements determine what a linear map does to basis elements, sufficient to determine what it does to any other element of a vector space.

Given basis $S_V = \{v_1, v_2, \ldots, v_n\} \subset V$ and basis $S_W = \{w_1, w_2, \ldots, w_m\} \subset W$, we define $S_{VW} \subseteq \mathcal{L}(V, W)$ as a basis for the vector space of all linear maps from V to W as follows:

Let $E_{ij} \in \mathcal{L}(V, W)$ be defined by

$$E_{ij}(v_k) = \delta_k^j w_i \qquad \begin{array}{l} i = 1, 2, \ldots, m \\ k, j = 1, 2, \ldots, n \end{array}$$

where $\delta_k^j = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$ denotes the <u>Kronecker</u> symbol. For any $A \in \mathcal{L}(V, W)$, with matrix representation $\tilde{A} = [a_{ij}]$,

$$\left(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} E_{ij} \right)(v_k) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} E_{ij}(v_k)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \cdot \delta_k^j \, w_i$$

$$= \sum_{i=1}^{m} a_{ik} \, w_i$$

$$= \mathbb{A}(v_k) \qquad k = 1, 2, \cdots, n$$

This shows that $\{ E_{ij}, \ i = 1, 2, \cdots, m, \ j = 1, 2, \cdots, n \} = S_{VW}$ is a basis for $\mathcal{L}(V, W)$.

11. <u>Invariants</u> (or similarity invariants)

Associated to every square matrix $\tilde{A}$ there is a <u>resolvent</u> matrix $(\lambda I - \tilde{A})^{-1}$ viewed as function of a complex variable λ <u>provided the inverse exists</u>. The set of λ for which $(\lambda I - \tilde{A})$ is <u>not</u> invertible ~~one~~ is precisely the set

$$\{ \lambda \in \mathbb{C} : \ \det(\lambda I - \tilde{A}) = 0 \}.$$

Now $\chi_{\tilde{A}}(\lambda) = \det(\lambda I - \tilde{A})$ is a <u>polynomial</u> in λ of degree n. It is a <u>monic</u> polynomial of the form:
(We call it the <u>characteristic</u> polynomial of $\tilde{A}$)

$$\chi_{\tilde{A}}(\lambda) = \lambda^n + p_{n-1}\lambda^{n-1} + p_{n-2}\lambda^{n-2} + \cdots + p_1\lambda + p_0$$

where the coefficients p_i depend on the matrix elements of $\tilde{A}$.

Suppose the same linear map A has two different matrix representations $\tilde{A}$ and $\tilde{A}'$ corresponding to two different choices of bases, as in section ~~8~~ 9) above. Then we can write

$$\tilde{A}' = P^{-1}\tilde{A}P.$$

<u>Caution</u>: the prime does <u>not</u> mean transpose here.

Then,

$$\begin{aligned}
\chi_{\tilde{A}'}(\lambda) &= \det(\lambda I - \tilde{A}') \\
&= \det(\lambda I - P^{-1}\tilde{A}P) \\
&= \det(\lambda P^{-1}P - P^{-1}\tilde{A}P) \\
&= \det(P^{-1}(\lambda I - \tilde{A})P) \\
&= \det(P^{-1})\det(\lambda I - \tilde{A})\det(P) \\
&= \det(\lambda I - \tilde{A}) \qquad \left(\text{since } \det(P^{-1}) = 1/\det(P)\right)
\end{aligned}$$

We thus see that the characteristic polynomial $\chi_{\tilde{A}}(\lambda)$ is a property of the underlying linear map and not dependent on a particular choice of basis.
It follows that the eigenvalues, i.e. roots of the characteristic polynomial, are invariant to change of basis (or a similarity invariant).

We call the set $\{\lambda : \det(\lambda I - \tilde{A}) = 0\}$ the spectrum of $\tilde{A}$. It can also be referred to as the spectrum of A the underlying linear map, from the discussion above.

The equation,

$$\tilde{A}x = \lambda x \qquad (*)$$

has a solution $x \neq 0$ for a given complex number λ, iff (if and only if),

$$\det(\lambda I - \tilde{A}) = 0$$

iff

$$\lambda \in \text{spectrum}(\tilde{A}).$$

In that case x is an eigenvector

corresponding to the eigenvalue λ.

<u>Verify</u>: If $\tilde{A}$ is a real matrix, then $\lambda \in$ spectrum $(\tilde{A})$ iff $\bar{\lambda} \in$ spectrum$(\tilde{A})$ where the <u>overbar</u> denotes complex conjugation.

<u>Verify</u>: If $\tilde{A}$ is real and λ ~~has~~ an eigenvalue of $\tilde{A}$ with $\mathrm{Im}(\lambda)$ = imaginary part of $\lambda \neq 0$, then for an eigenvector x corresponding to λ, $\mathrm{Im}(x) \neq 0$.

There are other, <u>related</u> invariants of $\tilde{A}$ that we can define. First, let trace $(\tilde{A})$ denoted by

$$tr(\tilde{A}) = \sum_{i=1}^{n} a_{ii}$$

be the sum of diagonal elements of $\tilde{A}$. Suppose $\tilde{B} = [b_{ij}]$ is another $n \times n$ matrix. Then, from the definition of trace,

$$tr(\tilde{A}\tilde{B}) = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki}$$

On the other hand,

$$\operatorname{tr}(\tilde{B}\tilde{A}) = \sum_{i=1}^{n} \sum_{k=1}^{n} b_{ik} a_{ki}$$

$$= \sum_{k=1}^{n} \sum_{i=1}^{n} b_{ki} a_{ik}$$

(swapping i and k)

$$= \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki}$$

(swapping order of sums)

$$= \operatorname{tr}(\tilde{A}\tilde{B}) .$$

It follows that

$$\operatorname{tr}(P^{-1}\tilde{A}P)$$

$$= \operatorname{tr}(\tilde{A}PP^{-1})$$

$$= \operatorname{tr}(\tilde{A})$$

Thus $\operatorname{tr}(\tilde{A})$ is a <u>similarity invariant</u>.

<u>verify</u>: For any positive integer k, $\operatorname{tr}(\tilde{A}^{k})$ is a <u>similarity invariant</u>)

12. Spectrum of $\tilde{A}$ and solving an associated differential equation:

Consider the ordinary vector differential equation,

$$(**) \qquad \dot{x} = \tilde{A} x$$

where $\tilde{A}$ is an $n \times n$ constant matrix. We are interested in solving $(**)$. Let us look for special solutions to $(**)$ of the form,

$$x(t) = e^{t\lambda} x_0$$

for some complex number λ and vector x_0. Necessarily, $x(0) = x_0$ = initial condition.

From the differential equation,

$$\lambda e^{t\lambda} x_0 = \tilde{A} e^{t\lambda} x_0 .$$

Canceling $e^{t\lambda}$ from both sides we get

$$(\lambda I - A) x_0 = 0$$

i.e. (λ, x_0) constitute an (eigenvalue, eigenvector) pair. Thus we see that the spectrum plays an important role in *dynamics*.

13. Inverse of a matrix

Given $\tilde{A} = [a_{ij}]$ an $n \times n$ matrix, we say that $\tilde{B} = [b_{ij}]$ another $n \times n$ matrix, is the inverse of $\tilde{A}$, denoted as $\tilde{A}^{-1}$ provided $\tilde{A}\,\tilde{B} = I$ the identity matrix. Equivalently

$$\sum_{k=1}^{n} a_{ik} b_{kj} = \delta_j^k,$$

the *Kronecker symbol*.

Observe that,

$$\det(\tilde{A}) = \sum_{k=1}^{n} a_{ik} C_{ik}$$

$$= \sum_{k=1}^{n} a_{ik} C_{ki}^{T}$$

where

$C = [C_{ik}]$ the matrix of cofactors of a_{ik} and C^{T} = transpose of C.

From the previous page, if $\det(\tilde{A}) \neq 0$,

$$\sum_{k=1}^{n} a_{ik} \frac{C^T_{ki}}{\det(\tilde{A})} = 1$$

Further, for $j \neq i$,

$$\sum_{k=1}^{n} a_{ik} C^T_{kj} = \sum_{k=1}^{n} a_{ik} C_{jk}$$

$$= 0,$$

since the expression on the right is simply the determinant of a matrix in which the j^{th} and i^{th} rows are same.

Setting $\tilde{B} = \tilde{C}^T / \det(\tilde{A})$, it follows that

$$\sum_{k=1}^{n} a_{ik} b_{kj} = \delta^i_j$$

$$\Longleftrightarrow \quad \tilde{A}\tilde{B} = \underline{I} .$$

Hence the inverse

$$(\tilde{A})^{-1} = \frac{1}{\det(\tilde{A})} C^T$$

$$= \frac{\text{transpose of matrix of cofactors}}{\text{determinant}}$$

We call C the <u>adjugate</u> of $\tilde{A}$