Lecture 1(a) ENEE 660 Fall 2010

We begin with some <u>very basic</u> <u>linear algebra</u>. The models and signals in this course will be associated with vector spaces and operations on them.

1 Given a <u>field</u> F of scalars, here F will be either $\mathbb{C}$ the set of complex numbers, or $\mathbb{R}$ the set of real numbers, a <u>vector space</u> V over F is a non-empty set with elements called vectors and two rules of combination vector addition and scalar multiplication, respectively denoted

$$+ : V \times V \to V$$
$$(v, w) \mapsto v + w$$

and
$$\cdot : F \times V \to V$$
$$(a, v) \mapsto a \cdot v \quad \text{or simply } av$$

obeying the <u>axioms</u>.

(i) $v + w = w + v$

(ii) $(v + w) + z = v + (w + z)$

(iii) there exists $0 \in V$ such that $0 + v = v \quad \forall v \in V$

(iv) for each $v \in V$ there exists $(-v) \in V$ such that
$$v + (-v) = 0$$

(v) $a(bv) = (ab)v$ where $a, b \in F$, $v \in V$

(vi) $a(v + w) = av + aw$ $a \in F$, $v, w \in V$

(vii) $(a+b)v = av + bv$ $\quad a, b \in F,\ v \in V$

(viii) $1 \cdot v = v$ $\quad 1 \in F$

Examples $V = \mathbb{R}^n = \{(v_1, v_2, \dots, v_n) : v_i \in \mathbb{R}\}$

$V$ = Polynomials of degree $\le k$ with complex coefficients

$V$ = set of all continuous functions from interval $[0, 1]$ to $\mathbb{R}$.

2 A mapping $L : V \to W$ from vector space $V$ to vector space $W$ over the same field $F$ is said to be linear if

$$L(av + b\tilde{v}) = a L(v) + b L(\tilde{v})$$

for all $a, b \in F$ and $v, \tilde{v} \in V$

Example $V$ = Set of all polynomials of degree $\le k$ in indeterminate $\lambda$ and complex coefficients.

$W = V$

$L : V \to W$ defined as $L = \frac{d}{d\lambda}$

Observe that the linear mappings, all taken together form a vector space $\mathcal{L}(V, W)$.

over the same field F.

<u>hint:</u> $L_1 + L_2$ is defined by

$$(L_1 + L_2)(v) = L_1(v) + L_2(v)$$

$aL$ is defined by

$$(aL)(v) = a\,L(v)$$

3. An indexed set of vectors $\{v_1, v_2, \ldots, v_k\}$ in a vector space V, is said to be linearly <u>independent</u> if

$$\sum_{i=1}^{k} \alpha_i v_i = 0 \quad \Rightarrow \quad \alpha_i = 0 \quad \forall\, i = 1, 2, \ldots, k$$

Suppose V has the ~~property~~ that there is at least one indexed subset $\{v_1, v_2, \ldots, v_k\} \subset V$ which is linearly independent <u>and</u> any other subset of cardinality $> k$ is <u>not</u> linearly independent. We then say that V is a vector space of <u>dimension</u> k, and call such a linearly independent set $\{v_1, v_2, \ldots, v_k\}$ a <u>basis</u> for V.

4. If $V$ is of dimension $k$ and $S_k = \{v_1, v_2, \dots, v_k\}$ is a basis for $V$, then any vector $v \in V$ can be expressed as a linear combination of the basis vectors given.

<u>proof</u> : $S_{k+1} = \{v_1, v_2, v_3, \dots, v_k, v\}$ is <u>not</u> linearly independent since Cardinality $(S) = k+1$. Hence there exist constants $\alpha_i$ not all zero such that

$$\sum_{i=1}^{k} \alpha_i v_i + \alpha_{k+1} v = 0$$

$\alpha_{k+1} \neq 0$ since $S_k$ is a basis by hypothesis. Hence

$$v = \sum_{i=1}^{k} \left(-\frac{\alpha_i}{\alpha_{k+1}}\right) v_i \qquad \blacksquare$$

A basis representation of the form

$$v = \sum_{i=1}^{k} \beta_i v_i$$

is unique, for a fixed basis. We have thus a one-to-one correspondence

$$v \leftrightarrow (\beta_1, \beta_2, \dots, \beta_k) \in F^k$$

called <u>coordinate</u> <u>representation</u>.

In the previous paragraph we have associated a simple <u>ordered</u> tuple (list) of numbers, $(\beta_1, \beta_2, \dots, \beta_k)$ uniquely with $v \in V$. We can be a bit more choosy. We could decide to write this list as a column matrix

$$\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix} \quad \text{or} \quad \text{as a row matrix } (\beta_1 \ \beta_2 \cdots \beta_k).$$

<u>Suppose we go with column representation</u>. Then each vector $v$ is associated uniquely with its coordinate column matrix

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{pmatrix} \in \mathbb{R}^k \text{ or } \mathbb{C}^k$$

This choice will govern how we proceed in (6) below.

5. <u>Matrix representation</u> of a linear map.

Let $A \in \mathcal{L}(V, W)$. Let $S_V = \{v_1, v_2, \dots, v_n\}$ and $S_W = \{w_1, w_2, \dots, w_m\}$ be bases for $V$ and $W$ respectively. With these fixed bases, we represent

$$A v_j = \sum_{i=1}^{m} a_{ij} w_i \qquad j = 1, 2, \dots, n.$$

The $m \times n$ matrix

$$\tilde{A} = [a_{ij}]$$

represents $A$ in the chosen bases.

6. <u>Change of coordinates</u> under a change of bases.

Given two distinct bases $S = \{v_1, v_2, \ldots, v_k\}$ and $S' = \{v_1', v_2', \ldots, v_k'\}$ for $V$, let us denote

$$v_j' = \sum_{i=1}^{k} t_{ij} v_i$$

(*) Thus the <u>columns</u> of the matrix

$$T = [t_{ij}]$$

are the coordinate representations of the <u>new</u> basis vectors in $S'$ in terms of the <u>old</u> basis vectors in $S$.

Suppose the new coordinates of a general vector $v \in V$ are given by the ordered list $(\beta_1', \beta_2', \ldots, \beta_k')$ viewed as a column matrix.

Then,

$$v = \sum_{j=1}^{k} \beta_j' v_j'$$

$$= \sum_{j=1}^{k} \beta_j' \sum_{i=1}^{k} t_{ij} v_i$$

$$= \sum_{i=1}^{k} \left( \sum_{j=1}^{k} t_{ij} \beta_j' \right) v_i$$

$$= \sum_{i=1}^{k} \beta_i v_i$$

where $\beta_i = \sum_{j=1}^{k} t_{ij} \beta_j'$ indicates that

(**) the <u>rows</u> of T are used to express the <u>old</u> coordinates $\beta_i$ in terms of the <u>new</u> coordinates $\beta_j'$.

<u>Remark</u> Compare (*) and (**). We can say (**) succintly as

$$\beta = T \beta'$$

or diagrammatically,

$$S \xrightarrow{T} S' \qquad \text{(transforming sets)}$$

$$\beta \xleftarrow{T} \beta' \qquad \text{(transforming coordinates)}$$

The matrix $T$ is necessarily invertible. <u>Prove this</u>.

7. Suppose $A \in \mathcal{L}(V, W)$ and $\dim(V) = n$, $\dim(W) = m$. Let $\tilde{A} = [a_{ij}]$ denote the $m \times n$ matrix representation of $A$ with respect to bases $S_V \subset V$ and $S_W \subset W$. If,

$$w = Av$$

and $\beta_v$ and $\beta_w$ represent the coordinates (as columns) of $v$ and $w$ respectively. Then

$$\beta_w = \tilde{A}\, \beta_v \qquad \text{(matrix-vector multiplication)}$$

<u>Verify this</u>. It justifies the column representation chosen in (4) above.

8. Change of matrix representation under change of basis.

Let $V$ and $W$ be vector spaces of dimension $n$ and $m$ respectively. Let $A \in \mathcal{L}(V, W)$. In a basis $S_V$ for $V$ and $S_W$ for $W$, this map has a matrix representation

$$\tilde{A} = [a_{ij}]$$

with $m$ rows and $n$ columns. Suppose the bases undergo transformations to new bases

$$S_V \xrightarrow{T_V} S_V'$$

and

$$S_W \xrightarrow{T_W} S_W'$$

where $T_V$ and $T_W$ are $n \times n$ and $m \times m$ invertible matrices, respectively. Observe that for any vector $v \in V$ with image $w = Av \in W$, the coordinate vectors are given in the original bases by

$$(\dagger) \qquad \beta_w = \tilde{A}\beta_v$$

From (6) we note that the coordinate vectors $\beta_w'$ and $\beta_v'$ in the new bases, are related to the coordinate vectors in the old bases by

$$\beta_v = T_V \beta_v'$$

$$\beta_w = T_w \beta_w' .$$

Substitution in the formula ($\dagger$), we get

$$T_W \beta_w' = \tilde{A} T_V \beta_v'$$

or equivalently

$$\beta_w' = T_W^{-1} \tilde{A} T_V \beta_v'$$

$$= \tilde{A}' \beta_v'$$

where, $\tilde{A}' = T_W^{-1} \tilde{A} T_V$

is the matrix representation of A

in the new bases.

9. Change of matrix representation of a linear <u>self-map</u>

$$A : V \to W = V$$

Since $W$ and $V$ are same, there is only one basis to change, and only one matrix defining basis change, namely

$$T_W = T_V = P \quad \text{say}.$$

Then,

$$\tilde{A}' = P^{-1} \tilde{A} P$$

a similarity transformation of the matrix $A$.

<u>Remark</u> We care about what aspects of a linear map representation are <u>invariant</u> to (arbitrary) changes of bases.