where $G(s) = C(sI - A)^{-1}B$,

$$M(s) = K(sI - (A - HC))^{-1}B$$

$$N(s) = K(sI - (A - HC))^{-1}H$$

In the usual discussion of state-space theory of observer-controller design, we make the following assumptions:

$[A, B, C]$ is a minimal triple that realises $G(s)$;

K is such that $(A - BK)$ has spectrum $\subseteq \mathbb{C}^-$;

H is such that $(A - HC)$ has spectrum $\subseteq \mathbb{C}^-$;

then, the compensator consisting of M, N and the unity feedback of ξ is itself stable. Choice of spectrum of $A - HC$ governs how ~~form~~ fast $\hat{x}(t)$ converges to $x(t)$, Choice of spectrum of $A - BK$ is a ~~short~~ specification of the closed loop system behavior.

It is desirable to carry out the whole design process in the frequency domain, skipping the stage of state space realization and finding K and H. We do this below, using Figure 3 as a guide.

$$Y(s) = G(s)\, u(s)$$

$$u(s) = V(s) - \xi(s)$$

$$\xi(s) = M(s)\, u(s) + N(s)\, Y(s)$$

SISO assumption

From here-on we consider the single-input single-output (SISO) case only, allowing easier algebra. In that case,

$$Y(s) = \frac{G(s)}{1 + M(s) + N(s)\, G(s)}\, V(s)$$

define the closed loop transfer function

$$G_c(s) = G(s) \Big/ \big(1 + M(s) + N(s)\, G(s)\big)$$

Let $G(s) = b(s)/a(s)$, as before,
$M(s) = m_u(s) / m_d(s)$ and,
$N(s) = n_u(s) / n_d(s)$, where
all transfer functions are rational strict, proper and reduced (i.e. free of common factors in the numerator-denominator expressions).
Also a, m_d, n_d are monic.
Then, the closed loop transfer function

$$G_c(s) = \frac{b/a}{1 + \frac{m_u}{m_d} + \frac{n_u}{n_d}\frac{b}{a}}$$

$$= \frac{m_d n_d b}{m_d n_d a + m_u n_d a + n_u m_d b}$$

Suppose there exist m_d, m_u, n_d, n_u such that

$$* \quad \boxed{m_u n_d a + n_u m_d b = m_d n_d m}$$

for a <u>specified</u> polynomial $m(s)$
$\deg(m(s)) \leq n-1$.

Then, clearly, canceling $m_d n_d$ from the numerator and denominator, we obtain,

$$G_c = \frac{b}{a+m}$$

Specifying m and solving (*) yields any closed loop poles we might want — i.e. $m(s)$ is a <u>design specification</u>.

Solving (*) is equivalent to solving

(**) $$\frac{m_u}{m_d} a + \frac{n_u}{n_d} b = m$$

$$\left(\Longleftrightarrow \quad M a + N b = m \right)$$

for strictly proper transfer functions M, N.

We solve (**) in <u>three steps</u>.

① Find $x(s)$, $y(s)$ $\deg(x) < \deg(b)$ $\deg(y) < \deg(a)$ such that

$$x(s)\, a(s) + y(s)\, b(s) \equiv 1 .$$

This is the BEZOUT step.

(2) select monic polynomial $\delta(s)$ with $\deg(\delta) = n$.

Define $f = x\, m\, \delta$

$g = y\, m\, \delta$

then

$$\frac{f}{\delta} a + \frac{g}{\delta} b = (xa + yb) m$$

$$= m$$

Thus $\frac{f}{\delta}$ and $\frac{g}{\delta}$ solve $(**)$

but fail to meet the strict properness conditions. In fact they are polynomials and not rational functions. We remedy this in the next step.

(3) Divide g by a to get

$$g = l a + g_0$$

$$\deg(g_0) < \deg(a) = n$$

Define $f_0 = f + l b$

Then $M = \frac{f_0}{\delta}$, $N = \frac{g_0}{\delta}$

solve (**), <u>and</u> respect the strict proper rationality conditions.

<u>proof</u> :

$$M a + N b$$

$$= \frac{(f_0 a + g_0 b)}{\delta}$$

$$= \frac{(f + lb)a + (g - la)b}{\delta}$$

$$= \frac{fa + gb}{\delta}$$

$$= \frac{(xa + yb) m \delta}{\delta}$$

$$= m$$

$$\deg(f_0 a) = \deg(m\delta - g_0 b)$$

$$\deg(f_0) = \deg(m\delta - g_0 b) - \deg(a)$$

$$= \deg(m\delta - g_0 b) - n$$

$\deg(g_0) \leq n-1$ and $\deg(b) \leq n-1$

$\Rightarrow \deg(g_0 b) \leq 2n-2$

$\deg(m) \leqq n-1$ and $\deg(\delta) = n$

$\Rightarrow \deg(m\delta) \leq 2n-1$

Thus $\deg(m\delta - g_0 b) \leq 2n-1$

Hence $\deg(f_0) \leq n-1$ ✓

$< n = \deg(\delta)$

Further $\deg(g_0) < \deg(a) = n = \deg(\delta)$

Thus $\frac{f_0}{\delta} \triangleq M$ and $\frac{g_0}{\delta} \triangleq N$ are both rational, strict, proper. $\boxed{}$

We have thus shown that there is an algorithm to produce desired $G_c = \dfrac{b}{a+m}$ in <u>three distinct steps</u>

Compensator Design by Bezout

(1) Solve BEZOUT using *repeated* Euclidean division to obtain $r_{p-1} = g.c.d.\ (a, b) \equiv 1$ and back-substitution.

(2) Select δ monic, $\deg(\delta) = n$ and define $f = x m \delta$; $g = y m \delta$.

(3) $\longrightarrow$ Divide g by a

$$g = l a + g_0$$

$$\deg(g_0) < \deg(a) = n$$

define

$$f_0 = f + l b$$

define

$$M = \frac{f_0}{\delta} \; ; \quad N = \frac{g_0}{\delta}$$

(4) STOP

In practice we would *specify*

(or get a *specification* from a customer)

that $m(s)$ with $\deg(m) \le n-1$

be such that

$$a(s) + m(s)$$

has all its roots in desired locations in $\mathbb{C}^-$.

and

we would *choose* $\delta(s)$ monic of $\deg = n$ such that all roots of δ are in desired locations in $\mathbb{C}^-$; this determines how fast the *observer* estimates the state

Example

$$G(s) = \frac{b(s)}{a(s)} = \frac{s+2}{s^3+3}$$

1. (i)

$$\underset{\substack{\uparrow \\ a}}{(s^3+3)} = s^2(s+2) + (-2s^2+3)$$

$$= s^2(s+2) + (-2s)(s+2) + 4s+3$$

$$= s^2(s+2) + (-2s)(s+2) + 4(s+2) - 5$$

$$= \underset{\substack{\uparrow \\ q_1}}{(s^2-2s+4)}\ \underset{\substack{\uparrow \\ b}}{(s+2)} \ - \underset{\substack{\uparrow \\ r_1}}{5}$$

(ii)

$$\underset{\substack{\uparrow \\ b}}{(s+2)} = \underset{\substack{\uparrow \\ q_2}}{\left(-\frac{2}{5} - \frac{1}{5}s\right)}\underset{\substack{\uparrow \\ r_1}}{(-5)} + \underset{\substack{\uparrow \\ r_2}}{0}$$

Thus $r_{p-1} = r_1 = -5$

$$r_1 = a - q_1 b$$

$$-5 = (1)(s^3+3) + (-s^2+2s-4)(s+2)$$

$$\Rightarrow 1 = \left(-\frac{1}{5}\right)(s^3+3) + \left(\frac{s^2}{5} - \frac{2}{5}s + \frac{4}{5}\right)(s+2)$$

$$= x(s)\,a(s) + y(s)\,b(s)$$

Thus
$$\left.\begin{aligned} x(s) &= -\frac{1}{5} \\ y(s) &= \frac{s^2}{5} - \frac{2s}{5} + \frac{4}{5} \end{aligned}\right\}$$

2. Suppose we want (or the customer wants) the closed loop characteristic polynomial to be

$$\begin{aligned} p(s) &= a(s) + m(s) \\ &= (s+1)^3 \\ &= s^3 + 3s^2 + 3s + 1 \end{aligned}$$

Then $m(s) = 3s^2 + 3s - 2$

Now select $\delta(s)$ monic $\deg(\delta) = 3$

→ <u>We choose</u> $\delta(s) = s^3 + 3$

Then $f = x\,m\,\delta = \left(-\frac{1}{5}\right)(3s^2+3s-2)(s^3+3)$

$g = y\,m\,\delta = \left(\frac{s^2}{5} - \frac{2}{5}s + \frac{4}{5}\right)(3s^2+3s-2)(s^3+3)$

<u>3.</u> $g = l a + g_0$

But a divides g exactly since $\delta = a = s^3+3$

So $g_0 = 0 \Rightarrow f_0 = f + l b$

$$= \overbrace{\left(-\frac{1}{5}\right)\left(3s^2+3s-2\right)\left(s^3+3\right)}^{f}$$

$$+ \underbrace{\left(\frac{s^2}{5} - \frac{2}{5}s + \frac{4}{5}\right)\left(3s^2+3s-2\right)}_{l} \underbrace{(s+2)}_{b}$$

$$\cancel{= \frac{s^2}{5} - \frac{2}{5}s + \frac{4}{5}}$$

$$= \frac{1}{5}\left(3s^2+3s-2\right)\left\{ \begin{array}{l} (s^2-2s+4)(s+2) \\ -(s^3+3) \end{array} \right\}$$

$$= \frac{1}{5}\left(3s^2+3s-2\right)\left(\begin{array}{l} \cancel{s^3} - \cancel{2s^2} + \cancel{4s} + \cancel{2s^2} - \cancel{4s} + 8 \\ - \cancel{s^3} - 3 \end{array} \right.$$

$$= \frac{1}{5}\left(3s^2+3s-2\right)\left(5\right)$$

$$= 3s^2+3s-2$$

Thus $\quad M = \dfrac{f_0}{\delta} = \dfrac{3s^2+3s-2}{s^3+3}$

$$N = \frac{g_0}{\delta} = 0$$