ENEE631 Fall 2001
Lecture-5

Image Transforms (1)

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Announcement in Last Class

- Revision on Grading
  - Homework/Labs 25%
  - Project 40%
  - Two in-class exams 30%
  - Class participation 5%
- TA – Mr. Guan-Ming Su
  - gmsu@glue.umd.edu
- Assignment-1

Review of Last Class

- Image Quantization
  - Optimal mean square quantizer
  - Uniform quantizer
  - Compandor and a few possible purposes
    - *approximate MMSE quantizer without iterative process*
    - *avoid unbalanced relative change*
    - *take contrast sensitiveness into account*
  - Visual quantization to reduce contour effects
    - *dithering & halftoning*
- Review of 1-D and 2-D signal and systems

Outline for Today

- Foundations for Image Transforms
  - Matrices
  - Unitary transform (and orthogonal transform)
  - (K-L transform)

Why Do Transforms?

- Fast computation
  - E.g., convolution vs. multiplication
- Conceptual insights for various image processing
  - E.g., spatial frequency info. (smooth, moderate change, fast change, etc.)
- Obtain transformed data as measurement
  - E.g., radiology images (medical and astrophysics)
  - Need inverse transform
  - May need to get assistance from other transforms
- For efficient storage and transmission
  - Pick a few "representatives" (basis)
  - Just store/send the "contribution" from each basis

A vector space consists of a set of vectors, a field of scalars, a vector addition operation, and a scalar multiplication operation.

Basis Vectors and Basis Images

- A basis for a vector space ~ a set of vectors
  - Linearly independent ~ $\Sigma a_i \underline{v}_i = \underline{0}$ if and only if all $a_i = 0$
  - Uniquely represent every vector in the space by their linear combination ~ $\Sigma b_i \underline{v}_i$ ("*spanning set*" $\{\underline{v}_i\}$)
- Orthonormal basis
  - Orthogonality ~ inner product $<\underline{x}, \underline{y}> = \underline{y}^{*T}\underline{x} = 0$
  - Normalized length ~ $\| \underline{x} \|^2 = <\underline{x}, \underline{x}> = \underline{x}^{*T}\underline{x} = 1$
- 2-D inner product
  - $<F, G> = \Sigma_m \Sigma_n f(m,n)\, g^*(m,n) = \underline{G}_1^{*T}\underline{F}_1$ (rewrite matrix into vector)
    - *!! Don't do* FG ~ *may not even be a valid operation for MxN matrices!*
- 2D Basis Matrices (Basis Images)
  - Represent any images of the same size as a linear combination of basis images

1-D Unitary Transform

- 1-D sequence $\{ x(0), x(1), \ldots, x(N-1) \}$ as a vector
  - $\underline{y} = A\,\underline{x}$ and A is invertible
- Unitary matrix ~ $A^{-1} = A^{*T}$
  - Denote A^{*T} as A^H ~ *"Hermitian"*
  - $\underline{x} = A^{-1}\underline{y} = A^{*T}\underline{y} = \Sigma \underline{a}_i^{*T} y(i)$
    - *Hermitian of row vectors of A form a set of basis vectors* $\underline{a}_i^{*T} = [a^*(i,0), \ldots, a^*(i,N-1)]^T$
- Orthogonal matrix ~ $A^{-1} = A^T$
  - Real-valued unitary matrix A is orthogonal
  - Row vectors of A form a set of basis vectors

Examples of Unitary Matrix

- Which is unitary or orthogonal?

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \qquad \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

inv(A1) = [2, –3; -1, 2]

Properties of 1-D Unitary Transform $y = A x$

- Determinant and all eigenvalues have unit magnitude
  - ➔ verify this!
- Energy Conservation
  - $\| \underline{y} \|^2 = \| \underline{x} \|^2$
    - $\| \underline{y} \|^2 = \| A\underline{x} \|^2 = (A\underline{x})^{*T} (A\underline{x}) = \underline{x}^{*T} A^{*T} A\, \underline{x} = \underline{x}^{*T} \underline{x} = \| \underline{x} \|^2$
- Rotation
  - A unitary transformation is a rotation of a vector in N-dimension space, i.e., a rotation of basis coordinates

Properties of 1-D Unitary Transform (cont'd)

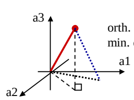
- Energy Compaction
  - Natural images generally distribute energy rather equally
    - *don't have too many fast transitions*
  - Many common unitary transforms tend to pack a large fraction of image energy into just a few transform coefficients
- Decorrelation
  - Highly correlated input elements ➔ quite uncorrelated output
  - Covariance matrix $E[\, (\underline{x} - E(\underline{x}))\, (\underline{x} - E(\underline{x}))^{*T}\,]$
    - small correlation implies small off-diagonal terms
- What unitary transform gives the best compaction and decorrelation?

2-D Linear Transforms

$$\begin{cases} y(k,l) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} x(m,n)\, a_{k,l}(m,n) \\ x(m,n) = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} y(k,l)\, a^*_{k,l}(m,n) \end{cases}$$

May generalize the transf. (series expansion) from NxN to NxM

- Image transform $\{a_{k,l}(m,n)\}$
  - y(k,l) is a transform coefficient for Image {x(m,n)}
  - {y(k,l)} is "Transformed Image"
  - Equiv to rewriting all from 2-D to 1-D and applying 1-D transform
- Orthonormality condition $\sum_{m=0}^{N-1} \sum_{n=0}^{N-1} a_{k,l}(m,n)\, a^*_{k',l'}(m,n) = \delta(k-k', l-l')$
  - assure any truncated expansion of the form $x_{P,Q}(m,n) = \sum_{k=0}^{P-1} \sum_{l=0}^{Q-1} y(k,l)\, a^*_{k,l}(m,n)$ will minimize sum of squared errors $\sigma_e^2 = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} [x(m,n) - x_{P,Q}(m,n)]^2$ when y(k,l) take values as above
- Completeness condition $\sum_{k=0}^{N-1} \sum_{l=0}^{N-1} a_{k,l}(m,n)\, a^*_{k,l}(m',n') = \delta(m-m', n-n')$
  - assure zero error when taking full basis



orth. proj. gives min. distance

2-D Transform (cont'd)

$$\begin{cases} y(k,l) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} x(m,n)\, a_{k,l}(m,n) \\ x(m,n) = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} y(k,l)\, a^*_{k,l}(m,n) \end{cases}$$

- Computational complexity
  - N^2 values to compute
  - N^2 terms in summation per output value
  - $O(N^4)$ for transforming an NxN image!

2-D Separable Unitary Transforms

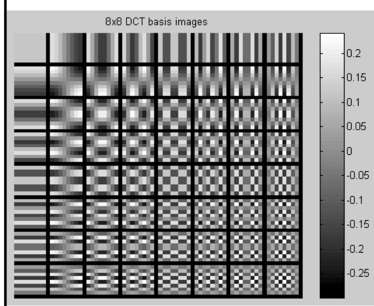
- Restrict to separable transform
  - $a_{k,l}(m,n) = a_k(m)\, b_l(n)$, denote this as $a(k,m)\, b(l,n)$
  - $\{a_k(m)\}_k$ and $\{b_l(n)\}_l$ are 1-D complete orthonormal sets of basis vectors
    - *use as row vectors to obtain unitary matrices $A=\{a(k,m)\}$ & $B=\{b(l,n)\}$*
  - Apply to columns and rows $Y = A\, X\, B^T$
    - often choose same unitary matrix as A and B *(e.g., 2-D DFT)*
- For square NxN image A: $Y = A\, X\, A^T \Leftrightarrow X = A^H\, Y\, A^*$
  - For rectangluar MxN image A: $Y = A_M\, X\, A_N^{\ T} \Leftrightarrow X = A_M^{\ H}\, Y\, A_N^{\ *}$
- Complexity ~ $O(N^3)$

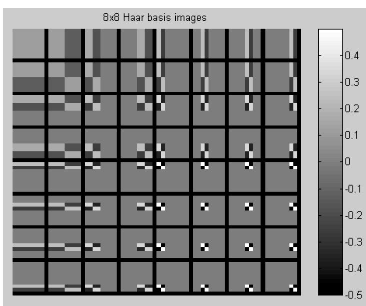
Basis Images

- $X = A^H\, Y\, A^* \Rightarrow x(m,n) = \Sigma_k\, \Sigma_l\; a^*(k,m)a^*(l,n)\; y(k,l)$
  - Represent X with NxN basis images weighted by coeff. Y
  - Obtain basis image by setting $Y=\{\delta(k-k_0,\, l-l_0)\}$ & getting X
    - $\{ a^*(k_0,m)a^*(l_0,n) \}_{m,n}$
    - *in matrix form* $A^*_{k,l} = a^*_k\, a_l^{*T}$ ~ a^*_k *is* k^{th} *column vector of* A^{*T}
    - *trasnf. coeff.* $y(k,l)$ *is the inner product of* $A^*_{k,l}$ *with the image*
- Example
  - Unitary transform or not? $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$ $X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
  - Find basis images
  - Represent an image X with basis images

(Jain's e.g.5.1, pp137)

Basis Images of Common Unitary Transform





See also: Jain's Fig.5.2 pp136

Summary

- Basis functions and basis images
  - Used for representing the space via their linear combinations
- Unitary transform
- Next time
  - Commonly used unitary transforms
    - *DFT, DCT, Haar*
    - *K-L transform*

Assignment

- Readings
  - Jain's book Sec.5.1-5.3, (5.11)

- Reminder: Assignment-1
  - Jasmine Lab hours with TA → Friday 2-5pm
    - *TA will be available for questions and discussions*
    - *PC camera will be available for completing II-7.*
  - Q&A