University of Maryland at College Park
Department of Electrical and Computer Engineering

ENEE 624 Advanced Digital Signal Processing

Problem Set 2 Solutions

Spring 2003

Monday, February 17, 2003

Problem 2.1

All three systems are linear, as each system is a cascade of linear systems.

System A: Let $v[n]$, $w[n]$, and $y[n]$ denote the outputs of the upsample-by-L system, the LTI system with transfer function $H(z)$, and the overall system output, respectively, when the input is a fixed but arbitrary signal $x[n]$. Then, when the system input is $x[n-k]$ (for some fixed but arbitrary integer k), the output of the upsample-by-L system is $v[n-kL]$, the output of the LTI system with transfer function $H(z)$ is $w[n-kL]$, and the output of the downsample-by-L system is $y[n-k]$. Hence system A is (linear and) time-invariant.

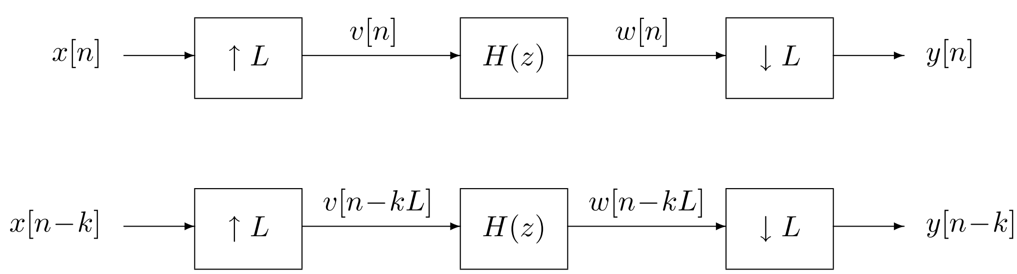

Figure 2.1-1 System A

We have

$$W(z) = H(z)\,V(z) = H(z)\,X(z^L)$$

and

$$\begin{aligned} Y(z) &= \frac{1}{L}\sum_{k=0}^{L-1} W\left(z^{1/L}W_L^k\right) = \frac{1}{L}\sum_{k=0}^{L-1} H\left(z^{1/L}W_L^k\right)\,X\left(\left(z^{1/L}W_L^k\right)^L\right) \\ &= \frac{1}{L}\sum_{k=0}^{L-1} H\left(z^{1/L}W_L^k\right)\,X\left(z\,W_L^{kL}\right) = X\,(z)\left[\frac{1}{L}\sum_{k=0}^{L-1} H\left(z^{1/L}W_L^k\right)\right] \end{aligned}$$

so the transfer function of System A is

$$H_{\mathrm{A}}(z) = \frac{1}{L}\sum_{k=0}^{L-1} H\left(z^{1/L}W_L^k\right) = H(z)\Big|_{\downarrow L} \;\;.$$

Consequently the impulse response of System A is given by $h_A[n] = h[Ln]$, and its frequency response via

$$H_{\mathrm{A}}\left(e^{j\omega}\right) = \frac{1}{L}\sum_{k=0}^{L-1} H\left(e^{j\frac{\omega-2k\pi}{L}}\right) \; .$$

System B: Clearly, if $L = 1$ or $H(z) = 0$, System B is LTI. So, assume $L > 1$ and $H(z) \neq 0$. Let $v[n]$, $w[n]$, and $y[n]$ denote the outputs of the downsample-by-L system, the LTI system with transfer function $H(z)$, and the overall system output, respectively, when the input is a fixed but arbitrary signal $x[n]$. We have (see Figure 2.1-2)

$$W(z) = H(z)\,V(z) = H(z)\left[\frac{1}{L}\sum_{k=0}^{L-1} X\left(z^{1/L}\,W_L^k\right)\right]$$

Finally,

$$Y(z) = W\left(z^L\right) = H\left(z^L\right)\left[\frac{1}{L}\sum_{k=0}^{L-1} X\left(z\,W_L^k\right)\right] = \frac{1}{L}H(z^L)\,X(z) + H\left(z^L\right)\left[\frac{1}{L}\sum_{k=1}^{L-1} X\left(z\,W_L^k\right)\right]$$

Hence, System B cannot LTI for $L > 2$ (unless $H(z) = 0$ for all z) since the response of the system at any $z = z_0$ depends on the input $X(z)$ at values of z different from z_0 (*e.g.*, $z = z_0\,W_L$). Since System B is linear, it cannot be time-invariant (unless $H(z) = 0$, or $L = 1$). However, System B *is* periodically time-varying. Specifically, given as input $x[n - kL]$ (for some fixed but arbitrary integer k), the output of the downsample-by-L system is $v[n - k]$, the corresponding output of the system with transfer function $H(z)$ is $w[n - k]$, and the corresponding overall system output is $y[n - kL]$.

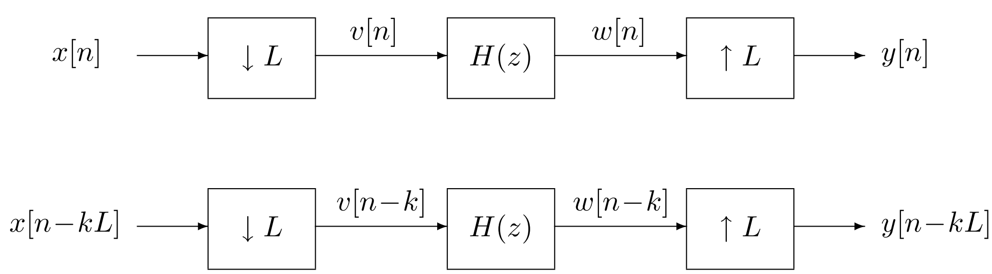

Figure 2.1-2 System B

System C: Let $v[n]$, $w[n]$, $s[n]$, $t[n]$ and $y[n]$ denote the outputs of the upsample-by-$2L$ system, the LTI system with transfer function $H(z)$, the downsample-by-L system, the LTI system with transfer function $G(z)$, and the overall system output, respectively, when the input is a fixed but arbitrary signal $x[n]$. Then, if the input is $x[n-k]$ (for some fixed but arbitrary integer k): the output of the upsample-by-$2L$ system is $v[n - 2kL]$; the output of the LTI system with transfer function $H(z)$ is $w[n - 2kL]$; the output of the downsample-by-L system $s[n-2k]$; the output of the LTI system with transfer function $G(z)$ is $t[n-2k]$, and the overall system output is $y[n - k]$. Hence the System C is (linear and) time invariant.

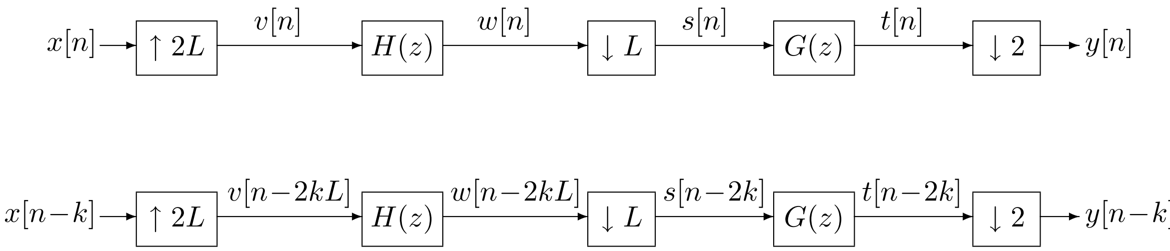

Figure 2.1-3 System C

We have

$$V(z) = X\left(z^{2L}\right) \qquad W(z) = V(z)\,H(z) = X\left(z^{2L}\right)H(z)$$

and

$$\begin{aligned} S(z) &= \frac{1}{L}\sum_{k=0}^{L-1} W\left(z^{1/L}W_L^k\right) = \frac{1}{L}\sum_{k=0}^{L-1} H\left(z^{1/L}W_L^k\right)\ X\left(\left(z^{1/L}W_L^k\right)^{2L}\right) \\ &= \frac{1}{L}\sum_{k=0}^{L-1} H\left(z^{1/L}W_L^k\right)\ X\left(z^2\,W_L^{2kL}\right) = X\left(z^2\right)\left[\frac{1}{L}\sum_{k=0}^{L-1} H\left(z^{1/L}W_L^k\right)\right]. \end{aligned}$$

In addition,

$$T(z) = S(z)\,G(z) = X\left(z^2\right)\left[\frac{1}{L}\sum_{k=0}^{L-1} G(z)\,H\left(z^{1/L}W_L^k\right)\right],$$

and

$$\begin{aligned} Y(z) &= \frac{1}{2}T\left(z^{1/2}\right) + \frac{1}{2}T\left(-z^{1/2}\right) \\ &= X(z)\left[\frac{1}{2L}G\left(z^{1/2}\right)\sum_{k=0}^{L-1} H\left(z^{1/(2L)}W_L^k\right) + \frac{1}{2L}G\left(-z^{1/2}\right)\sum_{k=0}^{L-1} H\left(W_{2L}\,z^{1/(2L)}W_L^k\right)\right]. \end{aligned}$$

Thus, the system function of System C is

$$H_{\mathrm{C}}(z) = \frac{1}{2L}\left[G\left(z^{1/2}\right)\sum_{k=0}^{L-1} H\left(z^{1/(2L)}W_L^k\right) + G\left(-z^{1/2}\right)\sum_{k=0}^{L-1} H\left(W_{2L}\,z^{1/(2L)}W_L^k\right)\right]$$

and its frequency response is

$$H_{\mathrm{C}}\left(e^{j\omega}\right) = \frac{1}{2L}\left[G\left(e^{j\omega/2}\right)\sum_{k=0}^{L-1} H\left(e^{j\frac{\omega-4k\pi}{2L}}\right) + G\left(-e^{j\omega/2}\right)\sum_{k=0}^{L-1} H\left(e^{j\frac{\omega-2\pi(2k+1)}{2L}}\right)\right]$$

Problem 2.2

We have

$$h_k[n] = h_0[n]\,\cos(2\pi k n/L) \quad \overset{\mathcal{Z}}{\longleftrightarrow} \quad H_k(z) = \frac{1}{2}\,H_0\Big(z\,W^{k}\Big) + \frac{1}{2}\,H_0\Big(z\,W^{-k}\Big)$$

where $W = e^{-j2\pi/L}$. Then

$$Y_0(z) = X\big(z^L\big)\;H_k(z) = X\big(z^L\big)\;\left[\frac{1}{2}\,H_0\Big(z\,W^{k}\Big) + \frac{1}{2}\,H_0\Big(z\,W^{-k}\Big)\right]\;.$$

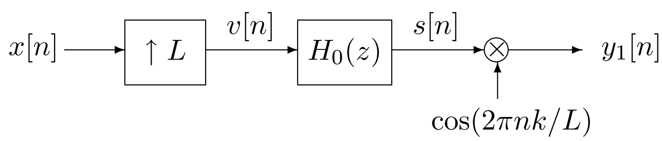

Figure 2.2-1 Right-hand side system

For the second system we have the following relationships

$$\begin{aligned}
S(z) &= X\big(z^L\big)\;H_0(z) \\
Y_1(z) &= \frac{1}{2}\,S\Big(z\,W^{k}\Big) + \frac{1}{2}\,S\Big(z\,W^{-k}\Big) \\
&= \frac{1}{2}\,X\left(\Big(z\,W^{k}\Big)^{L}\right)\;H_0\Big(z\,W^{k}\Big) + \frac{1}{2}\,X\left(\Big(z\,W^{-k}\Big)^{L}\right)\;H_0\Big(z\,W^{-k}\Big)\;.
\end{aligned}$$

Since $W^{kL} = W^{-kL} = 1$, we get

$$Y_1(z) = X\big(z^L\big)\;\left[\frac{1}{2}\,H_0\Big(z\,W^{k}\Big) + \frac{1}{2}\,H_0\Big(z\,W^{-k}\Big)\right] = Y_0(z)\;.$$

Next we turn to the example, with $L = 3$ and $k = 1$.

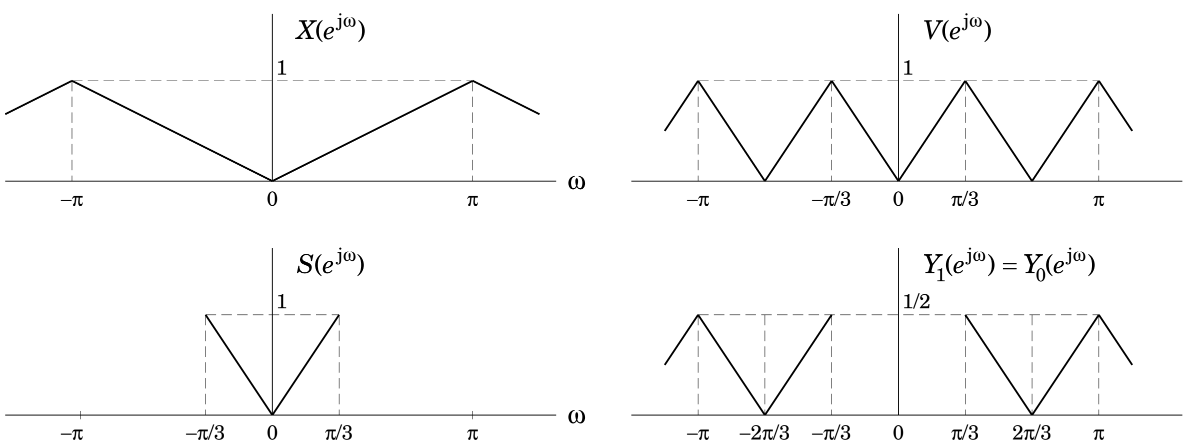

Figure 2.2-2

Problem 2.3

We have

$$\begin{aligned} Y_0(z) &= X(z)\left[\frac{1}{2}\,H_0\Big(z\,W^{k}\Big)+\frac{1}{2}\,H_0\Big(z\,W^{-k}\Big)\right] \\ Y_1(z) &= \frac{1}{2}\,X\Big(z\,W^{k}\Big)\;H_0\Big(z\,W^{k}\Big)+\frac{1}{2}\,X\Big(z\,W^{-k}\Big)\;H_0\Big(z\,W^{-k}\Big)\ . \end{aligned}$$

These are, in general, different. In fact, the system with response $y_0[n]$ is LTI, while the one with response $y_1[n]$ is not. For instance, let $k=1$, $L=2$. Let the input to both systems be $x[n]=e^{j\omega_o n}$ with $\omega_o=0$. We have

$$y_0[n]=C\,e^{j\omega_o n}=C\,,$$

where

$$C=\frac{1}{2}H_0\big(e^{j\omega_o}W\big)+\frac{1}{2}H_0\big(e^{j\omega_o}W^{-1}\big)=\frac{1}{2}H_0\big(e^{j\pi}\big)+\frac{1}{2}H_0\big(e^{-j\pi}\big)=H_0(-1)\ ,$$

while

$$y_1[n]=C_1\,e^{j(\omega_o-\pi)\,n}+C_2\,e^{j(\omega_o+\pi)\,n}=(C_1+C_2)\,(-1)^n\ .$$

Clearly, $y_0[n]$ is not equal to $y_1[n]$.

Problem 2.4

$$x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow y[n] = x[nM]$$

Figure 2.4-1 M-fold decimation system

Let $y[n]$ denote the response of the M-fold decimator system to an arbitrary N-periodic signal $x[n]$. Then, the response to $x[n-MN]$ is $y[n-N]$. But since $x[n]$ is N-periodic, we have $x[n] = x[n-MN]$. Hence $y[n-N] = y[n]$, *i.e.*, $y[n]$ is also periodic with period N. It only remains to determine as much as we can about the fundamental period of $y[n]$. We distinguish between the following two cases:

Case 1. M & N are relatively prime (*i.e.*, $gcd(M,N) = 1$):
We already know $y[n]$ is N-periodic, so we just need to check whether it is also periodic with some other period $N' < N$. We will assume that $y[n]$ is N'-periodic with $N' < N$ and arrive to a contradiction. Since $y[n]$ is also N-periodic, N must be an integer multiple of N', that is $N = N'p$ for some integer $p \geq 2$. For any integer n_o, let $A_y(n_o) \triangleq \{y[n_o], y[n_o+1], \cdots, y[n_o+N-1]\}$, and $A_x(n_o) \triangleq \{x[n_o], x[n_o+1], \cdots, x[n_o+N-1]\}$. Since M and N are relatively prime, the sets $A_y(0)$ and $A_x(0)$ are identical (although the ordering of the elements differs). More generally, for any integer n_o, the set $A_y(n_o)$ and $A_x(0)$ are identical. Furthermore, since $y[n] = y[n-N']$, the element sets $A_y(0)$, $A_y(N')$, $A_y(2N')$, $\cdots$, $A_y(pN')$ are all identical (also in order). This implies that $A_x(0)$, $A_x(MN')$, $A_x(2MN')$, $\cdots$, $A_x(MpN')$ are all identical and in order. This implies that each element in $A_x(0)$ occurs with multiplicity $k\,p$, where $k \geq 1$, and $p = N/N'$. Furthermore, we can readily see that $x[0] = x[kN'M]$ for any integer k; $x[M] = x[kN'M + M]$ for any integer k; etc, or due to $gcd(M,N) = 1$, $x[\ell] = x[kN'M + \ell]$ for all k and ℓ, which implies that the sequence $x[n]$ would have to be periodic with period $N'M$. So $N'M$ is a multiple of N. Since N and M are relatively prime, N' has to be a multiple of N, and thus cannot be less than N. Therefore N is the fundamental period of $y[n]$, *i.e.*, $L = N$.

Case 2. M & N are not relatively prime:
Then, there exists an integer $p \geq 2$, such that $M = pM_1$ and $N = pN_1$ where N_1 and M_1 are positive integers that are relatively prime. Specifically $p = \gcd(M,N) \geq 2$. If $y[n]$ denotes the response to a fixed but arbitrary N-periodic input $x[n]$, then, the response to $x[n-N_1M]$ is $y[n-N_1]$. But $N_1 M = N_1\,p\,M_1 = N\,M_1$, so $x[n-N_1M] = x[n-M_1N] = x[n]$, which implies that $y[n-N_1] = y[n]$, *i.e.*, N_1 is also a period of $y[n]$. It thus remains to show that N_1 is the fundamental. The proof is a replica of Case 1, as it relies on the fact that N_1 and M_1 are relatively prime. Assume there is an $N' < N_1$ such that $y[n-N'] = y[n]$ for all n. Then, since $y[n-N']$ is the response to the input $x[n-N'M]$ we have $x[n-N'M] = x[n]$, so $N'M$ is a multiple of N, *i.e.*, $N'\,p\,M_1$ is a multiple of $p\,N_1$. Equivalently, $N'\,M_1$ is a multiple of N_1. But that means that N' is a multiple of N_1 (since M_1 and N_1 are relatively prime). Therefore, the fundamental period of $y[n]$ is $L = N/gcd(M,N)$.

Problem 2.5

The overall system is shown in Figure 2.5-1.

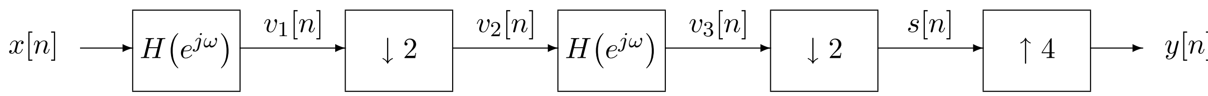

Figure 2.5-1 Overall system

The spectra of $v_1[n]$, $v_2[n]$, $v_3[n]$, and $s[n]$ are shown in Figure 2.5-2.

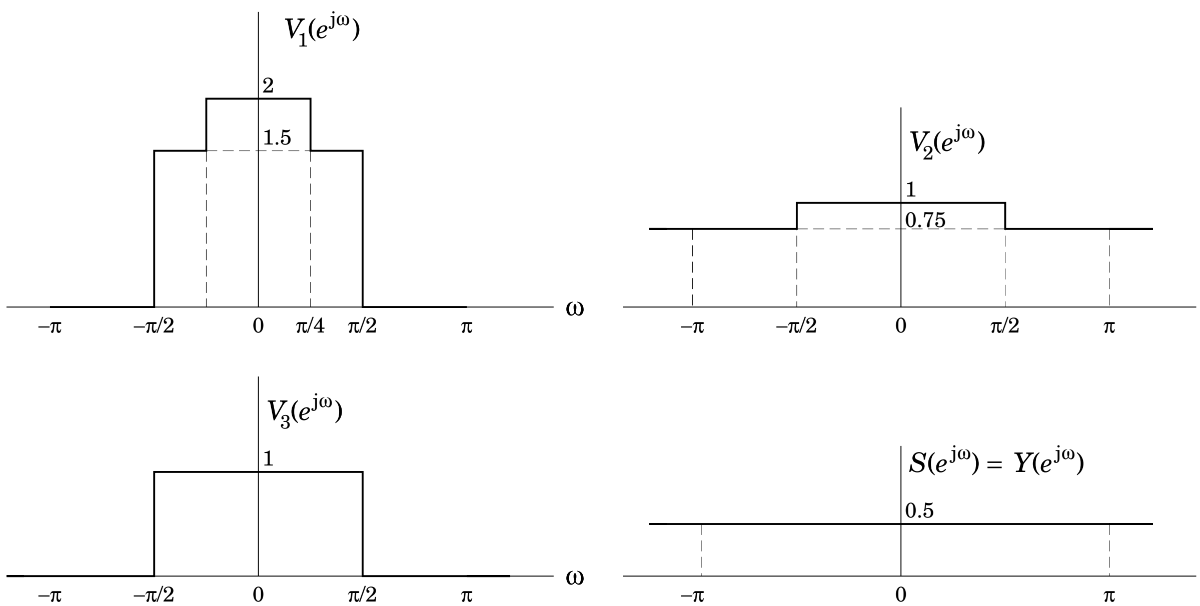

Figure 2.5-2

Therefore $s[n] = \frac{1}{2}\delta[n]$, which implies that $y[n] = \frac{1}{2}\delta[n] = s[n]$, and so $Y\left(e^{j\omega}\right) = S\left(e^{j\omega}\right) = \frac{1}{2}$ for all ω.

Problem 2.6

(a) We have

$$x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow \boxed{\downarrow M} \longrightarrow \boxed{\downarrow L} \longrightarrow \boxed{\uparrow M} \longrightarrow y[n]$$

$$x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow \boxed{\downarrow ML} \longrightarrow \boxed{\uparrow M} \longrightarrow y[n]$$

$$x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow \boxed{\downarrow L} \longrightarrow \boxed{\downarrow M} \longrightarrow \boxed{\uparrow M} \longrightarrow y[n]$$

$$x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow \boxed{\uparrow M} \longrightarrow y[n]$$

Figure 2.6-1 System for part (a)

(b) The system is of the form shown in Figure 2.6-2,

$$x[n] \rightarrow \boxed{\uparrow 3} \rightarrow \boxed{H_1(z)} \rightarrow \boxed{\uparrow 2} \rightarrow \boxed{H_2(z)} \rightarrow \boxed{\downarrow 5} \rightarrow \boxed{H_3(z)} \rightarrow \boxed{\downarrow 3} \rightarrow \boxed{H_4(z)} \rightarrow \boxed{\downarrow 2} \rightarrow y[n]$$

Figure 2.6-2 System for part (b)

(i) with

$$H_1(z) = z^{-1}, \quad H_2(z) = z^{-3}, \quad H_3(z) = z^{-2}, \quad H_4(z) = z^{-1};$$

(ii) or, equivalently, with

$$H_1(z) = 1, \quad H_2(z) = z^{-5}, \quad H_3(z) = z^{-2}, \quad H_4(z) = z^{-1};$$

(iii) or, equivalently, with

$$H_1(z) = 1; H_2(z) = 1, \quad H_3(z) = z^{-3}, \quad H_4(z) = z^{-1};$$

(iv) or, equivalently, with

$$H_1(z) = 1, \quad H_2(z) = 1, \quad H_3(z) = 1, \quad H_4(z) = z^{-2};$$

Thus, the system is also equivalent to those shown in Figure 2.6-3.

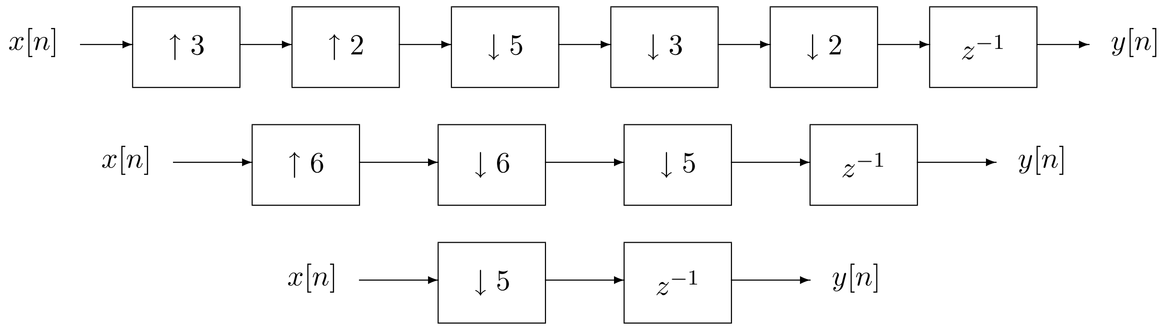

Figure 2.6-3 System for part (b)

(c) We have

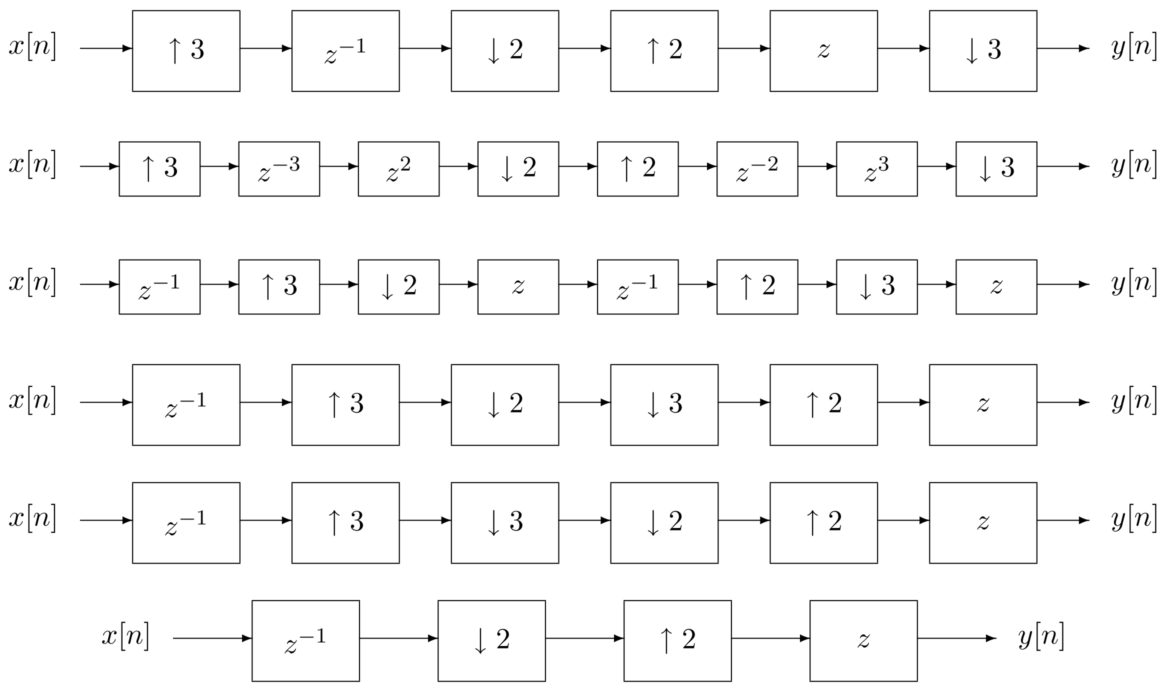

Figure 2.6-4 System for part (c)

Problem 2.7

(a) We have

$$\begin{aligned} G(z) &= \frac{1}{2}\,H\Big(z^{1/2}\Big) + \frac{1}{2}\,H\Big(-z^{1/2}\Big) \\ G\big(e^{j\omega}\big) &= \frac{1}{2}\,H\Big(e^{j\omega/2}\Big) + \frac{1}{2}\,H\Big(e^{j(\omega/2-\pi)}\Big) \end{aligned}$$

$H\big(e^{j\omega}\big)$ is allpass, *i.e.*, $H\big(e^{j\omega}\big) = d\,e^{j\phi(\omega)}$ for some $d > 0$. We would like $\phi(\omega)$ to be chosen such that

$$G\big(e^{j\omega}\big) = \frac{d}{2}\,e^{j\phi(\omega/2)} + \frac{d}{2}\,e^{j\phi(\omega/2-\pi)}$$

results in $\big|G\big(e^{j\omega}\big)\big| = c$. One way to achieve this is by selecting an $H\big(e^{j\omega}\big)$ that satisfies

$$\phi\Big(\frac{\omega}{2}\Big) = \phi\Big(\frac{\omega}{2} - \pi\Big) + \theta_o \qquad \text{for all } \omega\ .$$

Then,

$$G\big(e^{j\omega}\big) = \frac{d}{2}\,e^{j\phi(\omega/2)}\,\Big[1 + e^{-j\theta_o}\Big]$$

so that

$$|G\big(e^{j\omega}\big)\,| = \frac{d}{2}\sqrt{(1+\cos(\theta_o))^2 + \sin^2(\theta_o)}$$

which is independent of ω. Here's an example: set $\theta_o = 0$; then we need $\phi\big(\frac{\omega}{2}\big) = \phi\big(\frac{\omega}{2} - \pi\big)$. Clearly $\phi(\omega) = 0$ is one solution, yielding $G\big(e^{j\omega}\big) = d$, hence $g[n] = d\,h[n]$. Here is a more interesting example:

$$H\big(e^{j\omega}\big) = \begin{cases} 1 & \text{if } |\omega| < \pi/4, \text{ or } 3\pi/4 < |\omega| < \pi \\ -1 & \text{if } \pi/4 < |\omega| < 3\pi/4 \end{cases}$$

Then,

$$G\big(e^{j\omega}\big) = \begin{cases} 1 & \text{if } |\omega| < \pi/2 \\ -1 & \text{if } \pi/2 < |\omega| < \pi \end{cases}$$

which is clearly all pass.

(b) Let $v[n]$ be an allpass filter with $|V\big(e^{j\omega}\big)\,| = c$. Let $h[n]$ be obtained via upsampling $v[n]$ by 2. Then, $H\big(e^{j\omega}\big) = V\big(e^{j2\omega}\big)$, so $|H\big(e^{j\omega}\big)\,| = |V\big(e^{j2\omega}\big)\,| = c$. Then, let $g[n]$ be constructed by downsampling $h[n]$ by 2. By construction, $g[n] = v[n]$, so $g[n]$ is allpass. Let $V(z)$ be a rational function of order N. Clearly, $H(z) = V\big(z^2\big)$ is a rational function too (of order $2N$), and when $h[n]$ is decimated by 2, the resulting signal satisfies $g[n] = v[n]$, so $G(z)$ is allpass and rational. Hence the statement is false, even if $G(z)$ is restricted to be rational.