University of Maryland at College Park
Department of Electrical and Computer Engineering

ENEE 624 Advanced Digital Signal Processing

Problem Set 1 Solutions

Spring 2003

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Problem 1.1

(a) We have

$$Y(z)\,[z^{-1} - \frac{5}{2} + z] = X(z) \quad \Rightarrow \quad H(z) = \frac{1}{z^{-1} - \frac{5}{2} + z}$$

or, equivalently,

$$H(z) = \frac{z^{-1}}{1 - \frac{5}{2}z^{-1} + z^{-2}} \ .$$

Hence, $N = 2$, $a_1 = 1$, $a_j = 0$, for $j \neq 1$, $b_0 = 1$, $b_1 = -\frac{5}{2}$, $b_2 = 1$.

(b) Factoring the denominator of $H(z)$ we get

$$H(z) = \frac{z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - 2z^{-1}\right)} \ .$$

Therefore, $H(z)$ has poles at $p_1 = \frac{1}{2}$ and $p_2 = 2$, and zeros at $q_1 = \infty$ and $q_2 = 0$. Also

$$H(z) = \frac{-\frac{2}{3}}{1 - \frac{1}{2}z^{-1}} + \frac{\frac{2}{3}}{1 - 2z^{-1}} \ .$$

There are 3 possible regions of convergence. First, $\text{ROC}_1 = \{z : \ |z| > 2\}$, giving

$$\begin{aligned} h_1[n] &= -\frac{2}{3}\left(\frac{1}{2}\right)^n u[n] \ + \frac{2}{3}2^n u[n] \\ &= \frac{2}{3}\left[2^n - \frac{1}{2^n}\right] u[n] \ . \end{aligned}$$

This system is causal but not stable. Second, $\text{ROC}_2 = \left\{z : \ \frac{1}{2} < |z| < 2\right\}$, giving

$$h_2[n] \ = \ -\frac{2}{3}\left(\frac{1}{2}\right)^n u[n] \ - \frac{2}{3}2^n u[-n-1] \ .$$

This system is stable but not causal. Finally, $\text{ROC}_3 = \left\{z : \ |z| < \frac{1}{2}\right\}$, giving

$$\begin{aligned} h_3[n] &= \frac{2}{3}\left(\frac{1}{2}\right)^n u[-n-1] \ - \frac{2}{3}2^n u[-n-1] \\ &= \frac{2}{3}\left[\frac{1}{2^n} - 2^n\right] u[-n-1] \ . \end{aligned}$$

This system is neither stable nor causal.

Problem 1.2

If any of the systems S_1, S_2, S_3, S_4 were LTI it would satisfy

$$e^{j\pi n/3} \quad \longrightarrow \quad \lambda\, e^{j\pi n/3}$$

where λ is a complex number (*i.e.*, *independent* of n). We have

S_1:

$$\lambda\, e^{j\pi n/3} = n\, e^{j\pi n/3} \quad \Rightarrow \quad \lambda = n \ .$$

Thus S_1 is not LTI.

S_2:

$$\lambda\, e^{j\pi n/3} = e^{j7\pi n/3} = e^{j(2\pi n + \pi n/3)} = e^{j\pi n/3} \quad \Rightarrow \quad \lambda = 1 \ .$$

Thus S_2 *could be* LTI.

S_3:

$$\lambda\, e^{j\pi n/3} = e^{j\pi n/3}\, u[n] \quad \Rightarrow \quad \lambda = u[n] \ .$$

Thus S_3 is not LTI.

S_4:

$$\lambda\, e^{j\pi n/3} = e^{j\pi n/3} \left[\frac{1}{2\,j} - \frac{e^{-j2\pi n/3}}{2\,j} \right] \quad \Rightarrow \quad \lambda = \frac{1}{2\,j} - \frac{e^{-j2\pi n/3}}{2\,j} \ .$$

Thus S_4 is not LTI.

Problem 1.3

$$H(z) = K\,\frac{\left(1-q_1z^{-1}\right)\,\left(1-q_2z^{-1}\right)}{\left(1-p_1z^{-1}\right)\,\left(1-p_2z^{-1}\right)} = K\,\frac{\left(z-q_1\right)\,\left(z-q_2\right)}{\left(z-p_1\right)\,\left(z-p_2\right)}\;.$$

Let $q_i = r_i e^{j\theta_i}$, and $p_i = \beta_i e^{j\phi_i}$, with $r_i, \beta_i \geq 0$, for $i = 1, 2$. We also have

$$C(z) = H(z)\,H^*(1/z^*)\;.$$

So the zeros of $C(z)$ are at q_1, q_2, $1/q_1^*$ and $1/q_2^*$, and the poles at p_1, p_2, $1/p_1^*$ and $1/p_2^*$. Hence, each of $|q_1|$, $|q_2|$, $|p_1|$, snd $|p_2|$ are either 3/4 or 4/3. Since $H(z)$ is minimum-phase (*i.e.*, all poles and zeros are inside the unit circle), it must be that $|q_1| = |q_2| = |p_1| = |p_2| = 3/4$, *i.e.*, $q_i = \frac{3}{4}e^{j\theta_i}$, and $p_i = \frac{3}{4}e^{j\phi_i}$. From the figure, since $|H(e^{j\omega})|$ has local minima at $\omega = \pi/2$ and $\omega = -\pi/2$, the phases of the zeros of $H(z)$ are $\theta_1 = \pi/2$ and $\theta_2 = -\pi/2$, *i.e.*, $q_1 = 3j/4$ and $q_2 = -3j/4$. Similarly, since $|H(e^{j\omega})|$ has local maxima at $\omega = 0$ and $\omega = \pi$, the phases of the poles of $H(z)$ are $\phi_1 = 0$ and $\phi_2 = \pi$, *i.e.*, $p_1 = 3/4$ and $p_2 = -3/4$. It thus remains to determine K. From the figure, $|H(e^{j0})| = 1$ (0 dB). Hence,

$$|H(1)| = 1 \quad\Rightarrow\quad |K|\,\frac{|\,(1-3j/4)\;(1+3j/4)\,|}{|\,(1-3/4)\;(1+3/4)\,|} = 1 \quad\Rightarrow\quad |K| = \frac{7}{25}\;,$$

and since $h[n]$ is real-valued $K = \pm 7/25$. So the two possible minimum-phase systems consistent with the given information have transfer functions

$$H(z) = \frac{7}{25}\,\frac{\left(1-\frac{3j}{4}z^{-1}\right)\,\left(1+\frac{3j}{4}z^{-1}\right)}{\left(1-\frac{3}{4}z^{-1}\right)\,\left(1+\frac{3}{4}z^{-1}\right)}\;,$$

and

$$H(z) = -\frac{7}{25}\,\frac{\left(1-\frac{3j}{4}z^{-1}\right)\,\left(1+\frac{3j}{4}z^{-1}\right)}{\left(1-\frac{3}{4}z^{-1}\right)\,\left(1+\frac{3}{4}z^{-1}\right)}\;,$$

and both have $\mathrm{ROC}_H = \{z:\; |z| > 3/4\}$.

Problem 1.4

(a) Consider the cascade of two allpass systems with frequency response $H_1\left(e^{j\omega}\right)$ and $H_2\left(e^{j\omega}\right)$, respectively. Let $H\left(e^{j\omega}\right)$ denote the frequency response of the overall system. Since $H_1\left(e^{j\omega}\right)$ and $H_2\left(e^{j\omega}\right)$ are allpass, we have

$$|H_1\left(e^{j\omega}\right)| = c_1, \qquad \text{and} \qquad |H_2\left(e^{j\omega}\right)| = c_2.$$

This implies that

$$|H\left(e^{j\omega}\right)| = |H_1\left(e^{j\omega}\right)|\,|H_2\left(e^{j\omega}\right)| = c_1\, c_2\,,$$

which implies that the overall system is also allpass. Thus, the answer is no.

(b) Let $H_1\left(e^{j\omega}\right) = 1$ for all ω, and

$$H_2\left(e^{j\omega}\right) = \begin{cases} 1 & |\omega| < \pi/2 \\ -1 & \pi/2 < |\omega| \leq \pi \end{cases} .$$

Then, clearly, $|H_1\left(e^{j\omega}\right)| = |H_2\left(e^{j\omega}\right)| = 1$, *i.e.*, both $H_1\left(e^{j\omega}\right)$ and $H_2\left(e^{j\omega}\right)$ are allpass. Let $H\left(e^{j\omega}\right)$ denote the frequency response of the parallel interconection system. Then,

$$H\left(e^{j\omega}\right) = H_1\left(e^{j\omega}\right) + H_2\left(e^{j\omega}\right) = \begin{cases} 2 & |\omega| < \pi/2 \\ 0 & \pi/2 < |\omega| \leq \pi \end{cases} ,$$

which is clearly lowpass. Hence, the answer is yes.

(c)

$$H\left(e^{j\omega}\right) = c\,\frac{1 - \frac{1}{p^*}e^{-j\omega}}{1 - pe^{-j\omega}} = c\,\frac{-e^{-j\omega}}{p^*}\,\frac{1 - p^* e^{j\omega}}{1 - pe^{-j\omega}} = -c\,\frac{e^{-j\omega}}{p^*}\,\frac{\overline{1 - pe^{-j\omega}}}{1 - pe^{-j\omega}} .$$

Thus

$$|H\left(e^{j\omega}\right)| = |c|\,\frac{\left|e^{-j\omega}\right|}{|p^*|}\,\frac{\left|\overline{1 - pe^{-j\omega}}\right|}{|1 - pe^{-j\omega}|} = \frac{|c|}{|p|}\,,$$

i.e., independent of ω (and, therefore, allpass). In fact, via an analytic continuation argument one can show that, in light of the fact that $|H\left(e^{j\omega}\right)| = c_o$, if $H(z)$ has a pole at p, then it must have a zero at $1/p^*$.

(d) It is more convenient to consider the unity-gain term from $H(z)$ in part (c) [*i.e.*, neglect the constant-gain term c/p^*].

$$H_1\left(e^{j\omega}\right) = \frac{e^{-j\omega} - p^*}{1 - pe^{-j\omega}}\,,$$

or,

$$H_1\left(z\right) = \frac{z^{-1} - p^*}{1 - p\,z^{-1}}\,,$$

Note that if $p = 0$, $H_1(z) = z^{-1}$, and if $p = \infty$, $H_1(z) = z$, so that it covers all possible first-order factors. Hence, the general form is as follows:

$$H(z) = \prod_{i=1}^{N} \frac{z^{-1} - p_i^*}{1 - p_i\,z^{-1}} \qquad \text{for any } p_1,\, p_2,\, \cdots,\, p_N \in C \cup \{\infty\}.$$

Problem 1.5

(a) The condition to avoid aliasing in the band $(-\pi/T_1,\, \pi/T_1)$ passed by the reconstruction filter is $\Omega_s = 2\pi/T_1 \geq 2\,\Omega_o$, or, equivalently, in terms of T_1, $T_1 \leq \frac{\pi}{\Omega_o}$.

(b) We first note that $X_c(j\Omega)$ is nonzero and arbitrary only over the Ω range $\frac{2\Omega_o}{3} < |\Omega| < \Omega_o$. The width of this range is $2\Omega_o/3$, *i.e.*, considerably less than $2\Omega_o$. Note that by picking the reconstruction filter to have frequency response

$$H_s(j\Omega) = \begin{cases} K & \frac{2\Omega_o}{3} < |\Omega| < \Omega_o \\ 0 & \text{otherwise} \end{cases} ,$$

(where K is a gain that we will pick later), we effectively ensure that if $X_o(j\Omega) = X_c(j\Omega)$ for $\frac{2\Omega_o}{3} < |\Omega| < \Omega_o$, then $x_c(t) = x_o(t)$ (since both signals have no frequency components outside this range). Consequently, all we need to do is vary $\Omega_s = 2\pi/T_2$, from ∞ to 0, and observe for what values of Ω_s (or T_2) there is no aliasing in the bands of interest. Clearly, there is no aliasing for $\Omega_s \geq 2\Omega_o$ (case (a) in Fig. 1.5-2). Once we make, however, Ω_s just smaller than $2\,\Omega_o$ we start having aliasing. In particular, as Fig. 1.5-2 case (b) reveals, we have aliasing for $\frac{4}{3}\Omega_o < \Omega_s < 2\Omega_o$. However, there is no aliasing if $\Omega_s = \frac{4}{3}\Omega_o$. Furthermore, if $\Omega_o \leq \Omega_s \leq \frac{4}{3}\Omega_o$ there is no aliasing (case (c) in Fig. 1.5-2). Finally, although for $\frac{2}{3}\Omega_o < \Omega_s < \Omega_o$ there is aliasing (case (d) in Fig. 1.5-2), at $\frac{2}{3}\Omega_o = \Omega_s$ there is no aliasing (case (e) in Fig. 1.5-2). So we have no aliasing if and only if

$$\Omega_s \in \left\{ \left\{ \frac{2}{3}\Omega_o \right\} \cup \left[\Omega_o,\, \frac{4}{3}\Omega_o \right] \cup [2\Omega_o,\, \infty) \right\}$$

which is equivalent to

$$T_2 \in A \triangleq \left\{ \left(0,\, \frac{\pi}{\Omega_o} \right] \cup \left[\frac{3\,\pi}{2\,\Omega_o},\, \frac{2\,\pi}{\Omega_o} \right] \cup \left\{ \frac{3\,\pi}{\Omega_o} \right\} \right\} .$$

For any $T_2 \in A$ we have

$$x_o(t) = \frac{K}{T_2} x_c(t)$$

so by using a gain $K = T_2$ in the reconstruction filter $H_s(j\Omega)$ we get the desired $x_o(t) = x_c(t)$. Clearly, the largest T_2 that allows reconstruction of $x_c(t)$ from $x_2[n]$ is $T_2 = \frac{3\pi}{\Omega_o}$. The corresponding frequency response is shown in Figure 1.5-1.

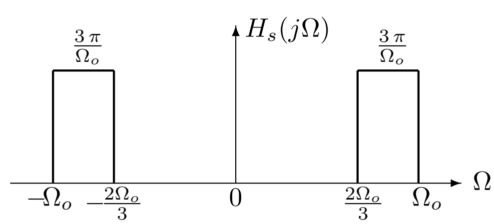

Figure 1.5-1 Frequency response $H_s(j\Omega)$

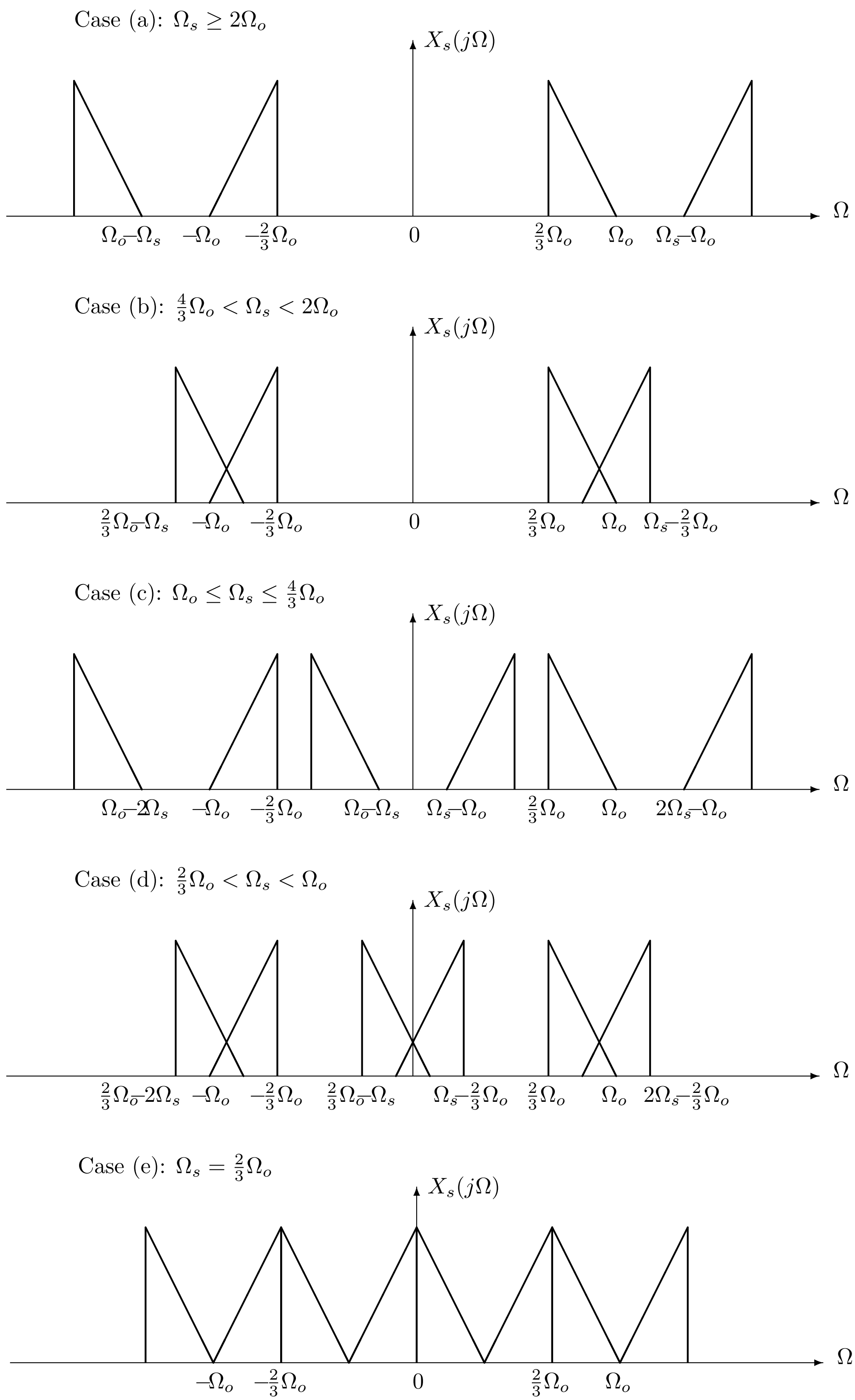

Figure 1.5-2 Fourier transform $X_s(j\Omega)$

Problem 1.6

Let

$$x[n] \longrightarrow y[n]\,, \qquad x_1[n] \longrightarrow y_1[n]\,, \qquad x_2[n] \longrightarrow y_2[n]\,. \tag{1}$$

First consider $x_1[n] = 2\,x[n]$. As the system possesses the additivity property,

$$x_1[n] = x[n] + x[n] \quad \longrightarrow \quad y_1[n] = y[n] + y[n]\,,$$

so

$$2\,x[n] \quad \longrightarrow \quad 2\,y[n]\,,$$

and by induction,

$$n\,x[n] \quad \longrightarrow \quad n\,y[n]\,, \qquad \text{for every positive integer } n. \tag{2}$$

Next, let $x_1[n] = 0 = 0 + 0 = x_1[n] + x_1[n]$. Then

$$x_1[n] = 0 = x_1[n] + x_1[n] \quad \longrightarrow \quad y_1[n] = y_1[n] + y_1[n] = 2y_1[n] \qquad \Rightarrow \qquad y_1[n] = 0\,,$$

This implies that

$$0 \quad \longrightarrow \quad 0\,, \tag{3}$$

which, when combined with (2), gives

$$n\,x[n] \quad \longrightarrow \quad n\,y[n]\,, \qquad \text{for every nonegative integer } n. \tag{4}$$

Next, let $x_1[n] = -x[n]$ and $x_2[n] = x_1[n] + x[n] = 0$. Due to (3) and the additivity property of the system we have

$$0 = x_2[n] = x_1[n] + x[n] \;\longrightarrow\; 0 = y_2[n] = y_1[n] + y[n] \qquad \Rightarrow \qquad y_1[n] = -y[n]\,.$$

So $-x[n] \longrightarrow -y[n]$, and by induction $-n\,x[n] \longrightarrow -n\,y[n]$, for every positive integer n, which, when combined with (4), gives

$$n\,x[n] \quad \longrightarrow \quad n\,y[n]\,, \qquad \text{for every integer } n. \tag{5}$$

Finally, let $x_1[n] = \frac{p}{q}\,x[n]$, where p, q are arbitrary integers (clearly, $q \neq 0$). Note that from (5) we have

$$p\,x[n] \quad \longrightarrow \quad p\,y[n]\,, \qquad \text{and} \qquad q\,x_1[n] \quad \longrightarrow \quad q\,y_1[n]\,.$$

However, $p\,x[n] = q\,x_1[n]$, so $p\,y[n] = q\,y_1[n]$ or, equivalently, $y_1[n] = \frac{p}{q}\,y[n]$, *i.e.*, $\frac{p}{q}\,x[n] \longrightarrow \frac{p}{q}\,y[n]$. Summarizing, if the system possesses the additivity property, then

$$x[n] \quad \longrightarrow \quad y[n]\,, \qquad \text{implies} \qquad \alpha\,x[n] \quad \longrightarrow \quad \alpha\,y[n]\,, \qquad \text{for any rational } \alpha.$$

It can be shown, however, that the system does not necessarily possess the scaling property for α irrational (in which case the system is not homogeneous). Construction of a system that satisfies additivity but not homogeneity is in fact nontrivial. One well-known example involves the use of Hamel bases as a means of describing such a system.

The interested reader may consult

Real Functions, by Casper Goffman, pp 150–151;

for details. In particular, the function $f(\cdot)$ defined at the end of the section on page 151 provides one such example of a memoryless system

$$x[n] \longrightarrow y[n] = f(x[n])\,,$$

where $f(\cdot)$ satisfies

$$f(x+y) = f(x) + f(y) \quad \text{but not} \quad f(\alpha\, x) = \alpha\, f(x) \quad \text{for all real } \alpha\ .$$

Roughly speaking, the function $f(\cdot)$ is defined via a Hamel basis according to which any real number is uniquely represented as a *finite* linear combination of the rational basis elements, *i.e.*, if

$$x_\alpha = \lambda_1\, x_{\alpha_1} + \lambda_2\, x_{\alpha_2} + \cdots + \lambda_M\, x_{\alpha_M}\ , \qquad \text{with basis elements } x_{\alpha_i}$$

then

$$f(x_\alpha) = \lambda_1\, f(x_{\alpha_1}) + \lambda_2\, f(x_{\alpha_2}) + \cdots + \lambda_M\, f(x_{\alpha_M})\,.$$

In particular, by picking the elements of the basis x_{α_1}, x_{α_2}, and setting $f(x_{\alpha_1}) = 1$ and $f(x_{\alpha_2}) = 0$, we obtain the desired function.