Bandpass Signal Representation

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All the radio frequency (RF) signals transmitted and received in wireless systems are bandpass signals. The spectrum of a bandpass signal is confined to a band not including 0 Hz in the frequency domain. An example of the Fourier transform of such a signal is shown in Figure 1. The frequency ω_c radians per second or $f_c = \omega_c/(2\pi)$ is called the *center frequency* for the signal and the signal bandwidth is $B = 2W$. When a low frequency message is intentionally translated up to a band centered around a frequency ω_c, this frequency is called the *carrier frequency*. For example, in ENEE 429w we will use the ISM (Instrumentation, Medical, and Scientific) band which extends from 902 to 928 MHz because an FCC license is not required to transmit in this band. The North American TDMA (Time-Division Multiple Access) standards IS-54 and TIA/EIA 136 specify signals with a bandwidth of $B = 30$ kHz and carrier frequencies in the band 869–894 MHz for base station to mobile station and 824–849 MHz for mobile to base station. GSM, the standard used extensively in Europe, uses a channel bandwidth of 200 MHz with carrier frequencies in the band 890–915 MHz for mobile to base station and 935–960 MHz for base to mobile station. In IS-95 CDMA (Code Division Multiple Access) systems, the channel bandwidth is 1.2288 MHz and the carrier bands are the same as for IS-54.

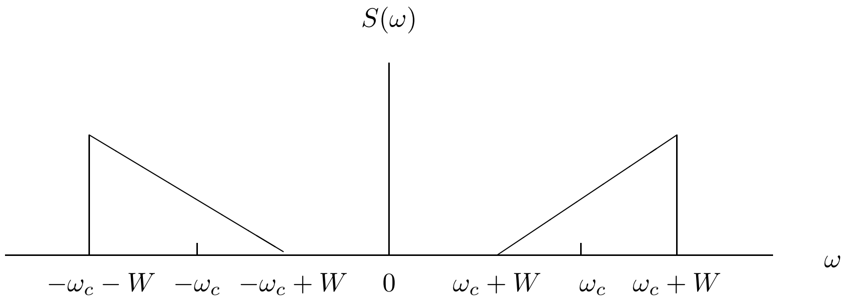

Figure 1: Example of the Fourier Transform of a Bandpass Signal

1 Hilbert Transforms

The Hilbert transform is a convenient tool to use in dealing with bandpass signals. The Hilbert transform of a signal $x(t)$ will be denoted by $\hat{x}(t)$ and is obtained by passing $x(t)$ through a filter with the transfer function

$$H(\omega) = -j\,\mathrm{sign}\,\omega = \begin{cases} -j & \text{for } \omega > 0 \\ 0 & \text{for } \omega = 0 \\ j & \text{for } \omega < 0 \end{cases} \tag{1}$$

This is illustrated in Figure 2. The Hilbert transform filter is an ideal 90° phase shifter. In the frequency domain

$$\hat{X}(\omega) = H(\omega)X(\omega) = -j\,\mathrm{sign}(\omega)X(\omega) \tag{2}$$

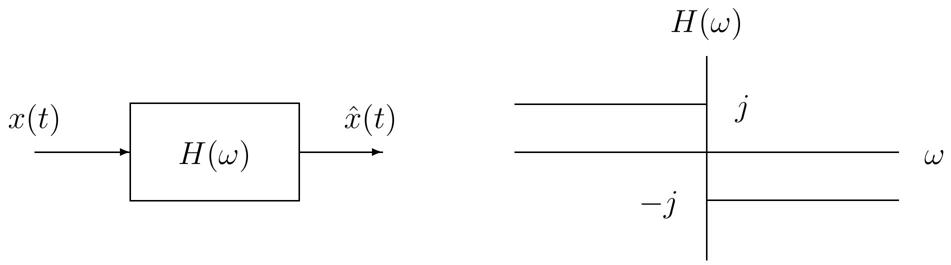

Figure 2: System for Forming Hilbert Transforms

It can be shown that the inverse Fourier transform of $H(\omega)$ is

$$h(t) = \frac{1}{\pi t} \tag{3}$$

Therefore, a signal and its Hilbert transform are related by the convolution integral

$$\hat{x}(t) = x(t) * \frac{1}{\pi t} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(\tau)}{t - \tau}\, d\tau \tag{4}$$

where $*$ represents convolution.

Example 1

Let $m(t)$ have the Fourier transform $M(\omega)$ that is identically zero for $|\omega| \geq W$. This signal is said to be *band limited* with cutoff frequency W. This lowpass signal is multiplied by the complex carrier signal $e^{j\omega_c t}$ with $\omega_c > W$ to form the bandpass signal

$$s(t) = m(t)e^{j\omega_c t} = m(t)\cos\omega_c t + j\, m(t)\sin\omega_c t \tag{5}$$

According to the frequency translation theorem,

$$S(\omega) = M(\omega - \omega_c) \tag{6}$$

Typical transforms for $M(\omega)$ and $S(\omega)$ are shown in Figure 3.

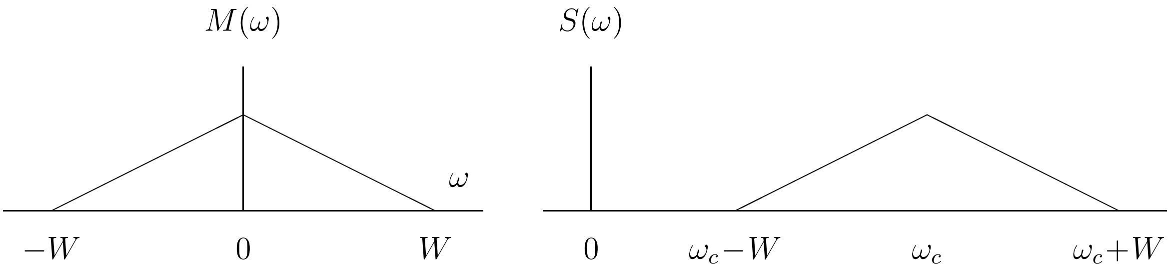

Figure 3: Sample Transforms

In this case

$$\hat{S}(\omega) = -j\,\mathrm{sign}(\omega) = -jM(\omega - \omega_c) \tag{7}$$

since $M(\omega - \omega_c) = 0$ for $\omega < 0$. In the time domain

$$\begin{aligned} \hat{s}(t) &= -j\, m(t) e^{j\omega_c t} = m(t) e^{j(\omega_c t - \frac{\pi}{2})} \\ &= m(t) \sin \omega_c t - j\, m(t) \cos \omega_c t \end{aligned} \tag{8}$$

Notice that the baseband message $m(t)$ remains unchanged but the carrier phase has been shifted by $-90°$.

1.1 Separation into Real and Imaginary Parts

Let $m_I(t)$ and $m_Q(t)$ be two real signals and define $m(t)$ to be the complex signal

$$m(t) = m_I(t) + j\, m_Q(t)$$

The signal $m_I(t)$ is called the *inphase* component of $m(t)$ and $m_Q(t)$ is called its *quadrature* component. The Hilbert transform of $m(t)$ is

$$\hat{m}(t) = m(t) * \frac{1}{\pi t} = m_I(t) * \frac{1}{\pi t} + j\, m_Q(t) * \frac{1}{\pi t} \tag{9}$$

Therefore,

$$\Re e\{m(t)\} \xrightarrow{\mathcal{H}} \Re e\{\hat{m}(t)\} \tag{10}$$

and

$$\Im m\{m(t)\} \xrightarrow{\mathcal{H}} \Im m\{\hat{m}(t)\} \tag{11}$$

Example 2

Let $m(t)$ be real and band limited with cutoff frequency W and let $\omega_c > W$. From (5), (8), (10), and (11), it follows that when $m(t)$ is real

$$m(t) \cos \omega_c t \xrightarrow{\mathcal{H}} m(t) \sin \omega_c t \tag{12}$$

and

$$m(t) \sin \omega_c t \xrightarrow{\mathcal{H}} -m(t \cos \omega_c t \tag{13}$$

Again notice that the lowpass message $m(t)$ remains unchanged but the carrier phase is shifted by $-90°$.

1.2 Some Useful Hilbert Transform Pairs

The following Hilbert transform pairs are included for reference. You should prove they are true.

1. $$\cos \omega_c t \stackrel{\mathcal{H}}{\Longrightarrow} \sin \omega_c t \tag{14}$$

2. $$\sin \omega_c t \stackrel{\mathcal{H}}{\Longrightarrow} -\cos \omega_c t \tag{15}$$

3. $$\cos(\omega_c t + \theta) \stackrel{\mathcal{H}}{\Longrightarrow} \cos\left(\omega_c t + \theta - \frac{\pi}{2}\right) \tag{16}$$

4. Let $m(t)$ be a lowpass signal with cutoff frequency W_1 and $c(t)$ a highpass signal with lower cutoff frequency $W_2 > W_1$. Then

$$m(t)c(t) \stackrel{\mathcal{H}}{\Longrightarrow} m(t)\hat{c}(t) \tag{17}$$

2 The Pre-Envelope or Analytic Signal

The *pre-envelope* or *analytic signal* associated with a bandpass signal $x(t)$ is defined to be

$$x_+(t) \stackrel{\triangle}{=} x(t) + j\,\hat{x}(t) \tag{18}$$

The Fourier transform of the pre-envelope is

$$X_+(\omega) = X(\omega) + j(-j\,\mathrm{sign}(\omega)X(\omega) = 2X(\omega)u(\omega) \tag{19}$$

where $u(\omega)$ is a unit step function. Notice that the transform of the pre-envelope is identically 0 for negative frequencies and double the original spectrum for positive frequencies. This is illustrated in Figure 4.

Example 3

Let $x(t) = \cos \omega_c t$. Then $\hat{x}(t) = \sin \omega_c t$. The Fourier transforms of these signals are

$$X(\omega) = \pi\delta(\omega - \omega_c) + \pi\delta(\omega + \omega_c)$$

and

$$\hat{X}(\omega) = \frac{\pi}{j}\delta(\omega - \omega_c) - \frac{\pi}{j}\delta(\omega + \omega_c)$$

The pre-envelope is

$$x_+(t) = \cos \omega_c t + j \sin \omega_c t = e^{j\omega_c t}$$

and

$$X_+(\omega) = 2\pi\delta(\omega - \omega_c)$$

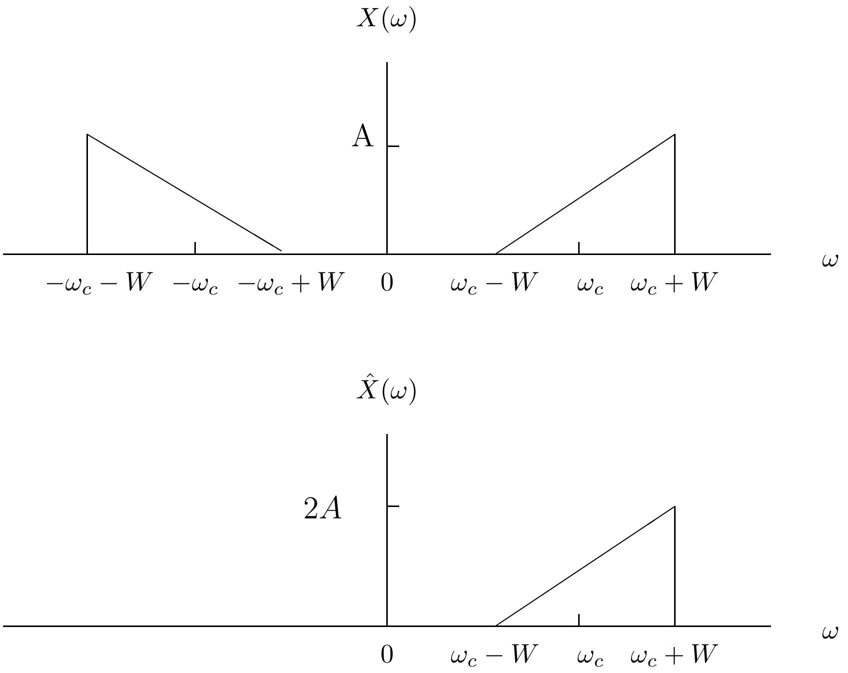

Figure 4: Frequency Domain Effect of Forming the Pre-Envelope

Notice that the real sinusoids have spectral lines at ω_c and $-\omega_c$ while the pre-envelope just has a line at ω_c.

▬

Example 4

Consider $x(t) = m(t)\cos\omega_c t$ as in Example 2. Then $\hat{x}(t) = m(t)\sin\omega_c t$ and the pre-envelope is

$$x_+(t) = x(t) + j\hat{x}(t) = m(t)\cos\omega_c t + j\,m(t)\sin\omega_c t = m(t)e^{j\omega_c t} \tag{20}$$

▬

3 The Complex Envelope

The *complex envelope* of a signal $x(t)$ is defined to be

$$\tilde{x}(t) = x_+(t)e^{-j\omega_c t} \tag{21}$$

According to the frequency translation theorem, the complex envelope has the transform

$$\tilde{X}(\omega) = X_+(\omega + \omega_c) = 2X(\omega + \omega_c)u(\omega + \omega_c) \tag{22}$$

Continuing the example shown in Figure 4, the effect of forming the complex envelope is shown in Figure 5. Notice that the complex envelope is a lowpass function with its spectrum located around the origin.

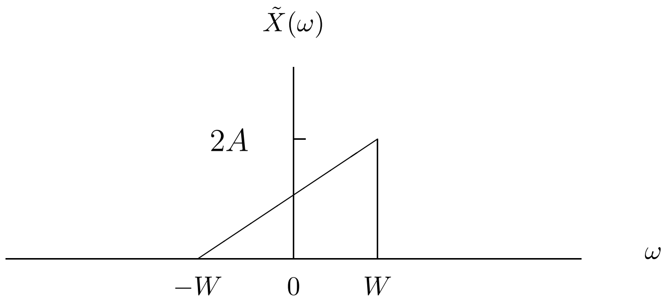

Figure 5: Spectrum of the Complex Envelope

Example 5

The complex envelope for the signal of Example 4 is

$$\tilde{x}(t) = m(t)e^{j\omega_c t}e^{-j\omega_c t} = m(t) \tag{23}$$

The signal of Example 4 can be used as a justification for using the term, envelope. The peaks of $x(t) = m(t)\cos\omega_c t$ pass through $\pm m(n\pi/\omega_c)$ at the positive and negative peaks of the carrier wave $\cos\omega_c t$. The signal $x(t)$ is called a *suppressed carrier, amplitude modulated* wave. The smooth curve $|m(t)|$ through the positive peaks of the wave is called the *real envelope* of the signal. The signal $m(t)$ can be positive or negative and retains the polarity information. More generally, $m(t)$ can be complex and contains amplitude and phase information. For this reason, it is called the complex envelope.

The real envelope of a signal is defined to be

$$a(t) = |m(t)| \tag{24}$$

4 Decomposition into Inphase and Quadrature Components

Suppose the complex envelope of $x(t)$ is

$$\tilde{x}(t) = x_I(t) + j\,x_Q(t) = x_+(t)e^{-j\omega_c t} \tag{25}$$

where $x_I(t) = \Re e\{x(t)\}$ is a real signal called the *inphase* component and $x_Q(t) = \Im m\{x(t)\}$ is a real signal called the *quadrature* component. Then

$$x_+(t) = \tilde{x}(t)e^{j\omega_c t} = [x_I(t) + j\,x_Q(t)][\cos\omega_c t + j\sin\omega_c t] \tag{26}$$

and

$$x(t) = \Re e\{x_+(t)\} = x_I(t)\cos\omega_c t - x_Q(t)\sin\omega_c t \tag{27}$$

Thus, it has been shown that any bandpass signal can be represented in terms of its inphase and quadrature baseband components.

The complex envelope has the polar form representation

$$\tilde{x}(t) = a(t)e^{j\theta(t)} \tag{28}$$

where

$$a(t) = |\tilde{x}(t)| \quad \text{and} \quad \theta(t) = \arg \tilde{x}(t)$$

With these definition, the pre-envelope can be expressed as

$$x_+(t) = a(t)e^{j[\omega_c t + \theta(t)]} \tag{29}$$

and

$$x(t) = a(t)\cos[\omega_c t + \theta(t)] \tag{30}$$

Therefore, we see that $a(t) = |\tilde{x}(t)|$ is the real envelope of the signal.

The real envelope is also the complex magnitude of the pre-envelope because

$$|\tilde{x}(t)| = |x_+(t)e^{-j\omega_c t}| = |x_+(t)|\,|e^{-j\omega_c t}| = |x_+(t)| \tag{31}$$

4.1 Finding the Inphase and Quadrature Components

According to the previous definitions,

$$\tilde{x}(t) = x_+(t)e^{-j\omega_c t} = [x(t) + j\,\hat{x}(t)]\,[\cos\omega_c t - j\sin\omega_c t] \tag{32}$$

Taking the real and imaginary parts gives

$$x_I(t) = \Re e\{\tilde{x}(t)\} = x(t)\cos\omega_c t + \hat{x}(t)\sin\omega_c t \tag{33}$$

and

$$x_Q(t) = \Im m\{\tilde{x}(t)\} = \hat{x}(t)\cos\omega_c t - x(t)\sin\omega_c t \tag{34}$$

An alternative method for finding the inphase and quadrature signals is shown in Figure 6. The bandpass signal $x(t)$ is multiplied by an inphase carrier $2\cos\omega_c t$ in the upper branch and by a quadrature carrier $-2\sin\omega_c t$ in the lower branch. The multipliers are often called *mixers* in radio systems. The *post-detection* filters $G(\omega)$ are ideal lowpass filters with a cutoff equal to the band limit of the baseband I and Q components. Using the bandpass signal representation of (27), the output of the upper mixer is

$$\begin{aligned} y_I(t) &= x(t)2\cos\omega_c t = x_I(t)2\cos^2\omega_c t - x_Q(t)2\sin\omega_c t\cos\omega_c t \\ &= x_I(t)(1+\cos 2\omega_c t) - x_Q(t)\sin 2\omega_c t \\ &= x_I(t) + x_I(t)\cos 2\omega_c t - x_Q(t)\sin 2\omega_c t \end{aligned} \tag{35}$$

Similarly, the output of the lower mixer is

$$\begin{aligned} y_Q(t) &= -x(t)2\sin\omega_c t = -x_I(t)2\sin\omega_c t\cos\omega_c t + x_Q(t)2\sin^2\omega_c t \\ &= -x_I(t)\sin 2\omega_c t) + x_Q(t)(1-\cos 2\omega_c t) \\ &= x_Q(t) - x_Q(t)\cos 2\omega_c t - x_I(t)\sin 2\omega_c t \end{aligned} \tag{36}$$

The terms multiplied by the sinusoids at frequency $2\omega_c$ have spectra centered around $2\omega_c$ and are eliminated by the post-detection filters.

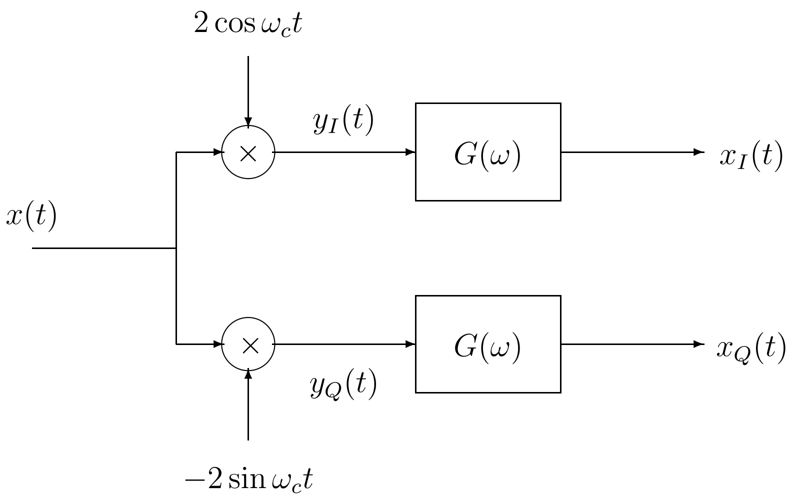

Figure 6: Using I-Q Demodulation to get the Inphase and Quadrature Component

4.2 Finding the Fourier Transforms of I and Q

The Fourier transforms of $x_I(t)$ and $x_Q(t)$ can be found directly from the transform of the complex envelope. Given a complex signal $s(t)$ with Fourier transform $S(\omega)$, it can be shown that

$$\overline{s(t)} \stackrel{\mathcal{F}}{\longleftrightarrow} \overline{S(-\omega)} \tag{37}$$

where the overbar indicates complex conjugate. Using this property and the linearity property of transforms, the following two identities result:

$$\begin{aligned}
\tilde{x}(t) &= x_I(t) + j\,x_Q(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \tilde{X}(\omega) = X_I(\omega) + j\,X_Q(\omega) && (38)\\
\overline{\tilde{x}(t)} &= x_I(t) - j\,x_Q(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \overline{\tilde{X}(-\omega)} = X_I(\omega) - j\,X_Q(\omega) && (39)
\end{aligned}$$

Adding (38) and (39) gives

$$X_I(\omega) = \frac{\tilde{X}(\omega) + \overline{\tilde{X}(-\omega))}}{2} \tag{40}$$

Subtracting gives

$$X_Q(\omega) = \frac{\tilde{X}(\omega) - \overline{\tilde{X}(-\omega))}}{2j} \tag{41}$$

As a special case, suppose the Fourier transform of the the bandpass signal has complex conjugate symmetry about the carrier frequency, that is,

$$X(\omega_c - w) = \overline{X(\omega_c + \omega)}$$

which for the complex envelope becomes

$$\tilde{X}(-\omega) = \overline{\tilde{X}(\omega)}$$

Then, according to (41), the quadrature component is identically zero and $X_I(\omega) = \tilde{X}(\omega)$. Similarly, if the bandpass transform has negative complex conjugate symmetry about ω_c, the inphase component is zero.

Example 6

As an example, consider the signal shown in Figure 5. The transforms of the inphase and quadrature components are shown in Figure 7.

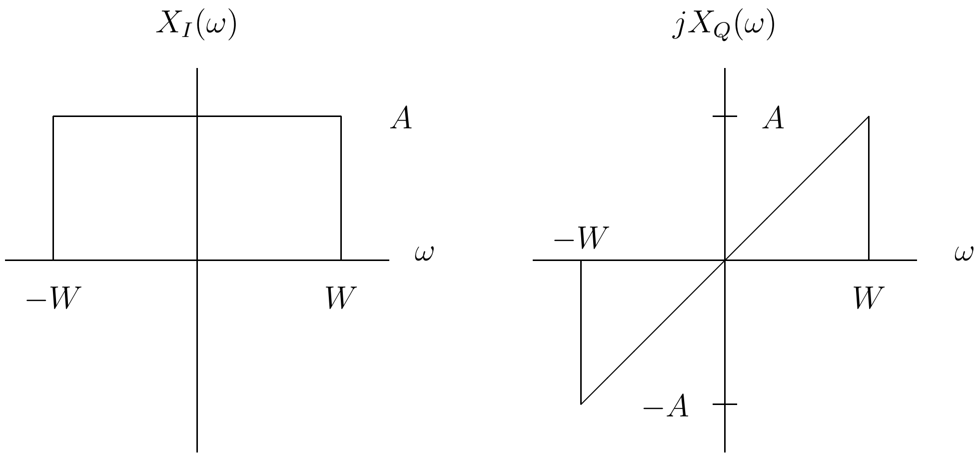

Figure 7: Example of Finding the Transforms of the I and Q Components

5 Bandpass Filters

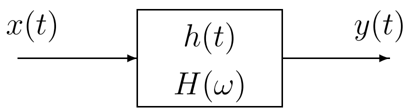

Figure 8: System Block Diagram

The complex signal representation is also convenient to use when analyzing the response of bandpass filters to bandpass signals. Let $x(t)$ in Figure 8 be a bandpass filter with center frequency ω_c. Assume the filter specified by the impulse response, $h(t)$, or transfer function $H(\omega)$ also has bandpass characteristics with the same center frequency. Then the transform of the filter output is

$$Y(\omega) = X(\omega)H(\omega) \tag{42}$$

and the pre-envelope of the output is

$$\begin{aligned} Y_+(\omega) &= 2Y(\omega)u(\omega) = 2X(\omega)H(\omega)u(\omega) \\ &= \frac{1}{2}[2X(\omega)u(\omega)]\,[2H(\omega)u(\omega)] \end{aligned}$$

$$= \frac{1}{2} X_+(\omega) H_+(\omega) \tag{43}$$

The transform of the complex envelope of the output is

$$\begin{aligned} \tilde{Y}(\omega) &= Y_+(\omega + \omega_c) \\ &= \frac{1}{2} X_+(\omega + \omega_c) H_+(\omega + \omega_c) = \frac{1}{2} \tilde{X}(\omega) \tilde{H}(\omega) \end{aligned} \tag{44}$$

Therefore, the complex envelope can be computed by the following convolution:

$$\begin{aligned} \tilde{y}(t) &= \frac{1}{2} \tilde{x}(t) * \tilde{h}(t) = \frac{1}{2} [x_I(t) + j\, x_Q(t)] * [h_I(t) + j\, h_Q(t)] \\ &= \frac{1}{2} [x_I(t) * h_I(t) - x_Q(t) * h_Q(t)] + j\, \frac{1}{2} [x_I(t) * h_Q(t) + x_Q(t) * h_I(t)] \end{aligned} \tag{45}$$

The inphase and quadrature components of the input and impulse response are lowpass signals that are slowly varying relative to the carrier signal. Therefore, it is convenient to use the complex envelopes in simulation and modelling of bandpass systems. The actual passband signal can be computed as

$$y(t) = \Re e \left\{ \tilde{y}(t) e^{j\omega_c t} \right\} \tag{46}$$

Example 7

Consider the ideal bandpass filter, $H(\omega)$, with transfer function

$$H(\omega) = \begin{cases} 1 & \text{for } \ \omega_c - \frac{B}{2} < \omega < \omega_c + \frac{B}{2} \\ 1 & \text{for } \ -\omega_c - \frac{B}{2} < \omega < -\omega_c + \frac{B}{2} \\ 0 & \text{elsewhere} \end{cases} \tag{47}$$

Then

$$\tilde{H}(\omega) = \begin{cases} 2 & \text{for } \ |\omega| < \frac{B}{2} \\ 0 & \text{elsewhere} \end{cases} \tag{48}$$

and

$$\tilde{h}(t) = \frac{1}{2\pi} \int_{-B/2}^{B/2} 2 e^{j\omega t}\, d\omega = \frac{2}{\pi} \frac{\sin \frac{B}{2} t}{t} \tag{49}$$

Let the input be $x(t) = u(t) \cos \omega_c t$ with $\omega_c >> 0$. Then to a good approximation, $\hat{x}(t) = u(t) \sin \omega_c t$. The pre-envelope is

$$x_+(t) = x(t) + j\, \hat{x}(t) = u(t) e^{j\omega_c t} \tag{50}$$

and the complex envelope is

$$\tilde{x}(t) = x_+(t) e^{-j\omega_c t} = u(t) \tag{51}$$

The complex envelope of the filter output is

$$\tilde{y}(t) = \frac{1}{2} \left(\frac{2}{\pi} \frac{\sin \frac{B}{2} t}{t} \right) * u(t) = \int_{-\infty}^{t} \frac{1}{\pi} \frac{\sin \frac{B}{2} \tau}{\tau}\, d\tau \tag{52}$$

Clearly, $\lim_{t\to-\infty} \tilde{y}(t) = 0$. The values for $t = 0$ and ∞ can be found by using the identity

$$\tilde{y}(\infty) = \frac{1}{2}\int_{-\infty}^{\infty} \tilde{h}(\tau)\, d\tau = \frac{1}{2}\tilde{H}(0) = 1 \tag{53}$$

Observing that $\tilde{h}(t)$ is an even function, it then follows that

$$\tilde{y}(0) = \frac{1}{2}\int_{-\infty}^{0} \tilde{h}(\tau)\, d\tau = \frac{1}{2}\tilde{y}(\infty) = \frac{1}{2} \tag{54}$$

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