

Secret Key and Private Key Constructions for Simple Multiterminal Source Models

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Abstract—This work is motivated by recent results of Csiszár and Narayan (*IEEE Trans. on Inform. Theory*, Dec. 2004), which highlight innate connections between secrecy generation by multiple terminals and multiterminal Slepian-Wolf near-lossless data compression (sans secrecy restrictions). We propose a new approach for constructing secret and private keys based on the long-known Slepian-Wolf code for sources connected by a virtual additive noise channel, due to Wyner (*IEEE Trans. on Inform. Theory*, Jan. 1974). Explicit procedures for such constructions, and their substantiation, are provided.

I. INTRODUCTION

The problem of secret key generation by multiple terminals, based on their observations of distinct correlated signals followed by public communication among themselves, has been investigated by several authors ([9], [1], among others). It has been shown that these terminals can generate common randomness which is kept secret from an eavesdropper privy to the public interterminal communication. Of particular relevance to us are recent results in [5] for models with an arbitrary number of terminals, each of which observes a distinct component of a discrete memoryless multiple source (DMMS). Unrestricted public communication is allowed between these terminals. All the transmissions are observed by all the terminals and by the eavesdropper. Two models considered in [5] are directly relevant to our work, and these are first briefly described below.

(i) Suppose that $m \geq 2$ terminals observe n i.i.d. repetitions of the random variables (rvs) X_1, \dots, X_m , denoted by $\mathbf{X}_1, \dots, \mathbf{X}_m$, respectively. A secret key (SK) generated by these terminals consists of “common randomness,” based on public interterminal communication, which is concealed from an eavesdropper with access to this communication. The largest (entropy) rate of such a SK is termed the SK-capacity, denoted by C_{SK} , and is shown in [5] to equal

$$C_{SK} = H(X_1, \dots, X_m) - R_{min}, \quad (1)$$

where

$$R_{min} = \min_{(R_1, \dots, R_m) \in \mathcal{R}} \sum_{i=1}^m R_i,$$

with

$$\mathcal{R} = \{(R_1, \dots, R_m) : \sum_{i \in B} R_i \geq H(\{X_j, j \in B\} | \{X_j, j \in B^c\}), B \subset \{1, \dots, m\}\},$$

where $B^c = \{1, \dots, m\} \setminus B$.

(ii) For a given subset $A \subset \{1, \dots, m\}$, a private key (PK) for the terminals in A , private from the terminals in A^c , is a SK generated by the terminals in A (with the possible help of the terminals in A^c), which is concealed from an eavesdropper with access to the public interterminal communication and also from the “helper” terminals in A^c (and, hence, private). The largest (entropy) rate of such a PK is termed the PK-capacity, denoted by $C_{PK}(A)$. It is shown in [5] that

$$C_{PK}(A) = H(\{X_i, i \in A\} | \{X_i, i \in A^c\}) - R_{min}(A), \quad (2)$$

where

$$R_{min}(A) = \min_{\{R_i, i \in A\} \in \mathcal{R}(A)} \sum_{i \in A} R_i,$$

with

$$\mathcal{R}(A) = \{\{R_i, i \in A\} : \sum_{i \in B} R_i \geq H(\{X_j, j \in B\} | \{X_j, j \in B^c\}), B \subset A\}.$$

The results above afford the following interpretation. The SK-capacity C_{SK} , i.e., largest rate at which all the m terminals can generate a SK, is obtained by subtracting from the maximum rate of shared common randomness achievable by these terminals, viz. $H(X_1, \dots, X_m)$, the smallest sum-rate R_{min} of the data-compressed interterminal communication which enables each of the terminals to acquire this maximal common randomness. A similar interpretation holds for the PK-capacity $C_{PK}(A)$ as well, with the difference that the terminals in A^c , which act as helpers but must not be privy to the secrecy generated, can simply “reveal” their observations. Hence, the entropy terms in (1) are now replaced in (2) with additional conditioning on $\{X_i, i \in A^c\}$. It should be noted that R_{min} and $R_{min}(A)$ are obtained as solutions to Slepian-Wolf (SW) multiterminal near-lossless data compression problems *not involving any secrecy constraints*. This characterization

of the SK-capacity and PK-capacity in terms of the decompositions above also mirrors the consecutive stages in the random coding arguments used in establishing these results. For instance, and loosely speaking, to generate a SK, the m terminals first generate common randomness (without any secrecy restrictions), say a rv L of entropy rate $\frac{1}{n}H(L) > 0$, through SW-compressed interterminal communication \mathbf{F} . This means that all the m terminals acquire the rv L with probability $\cong 1$. The next step entails an extraction from L of a SK $K = g(L)$ of entropy rate $\frac{1}{n}H(L|\mathbf{F})$, by means of a suitable operation performed *identically* at each terminal on the acquired common randomness L . When the common randomness first acquired by the m terminals is maximal, i.e., $L = (\mathbf{X}_1, \dots, \mathbf{X}_m)$ with probability $\cong 1$, then the corresponding SK $K = g(L)$ has the best rate C_{SK} given by (1). A similar approach is used to generate a PK of rate given by (2).

The discussion above suggests that techniques for multiterminal SW data compression could be used for the *construction* of SKs and PKs. Next, in SW coding, the existence of linear data compression codes with rates arbitrarily close to the SW bound has been long known [3]. In particular, when the i.i.d. sequences observed at the terminals are related to each other through virtual communication channels characterized by independent additive noises, such linear data compression codes can be obtained in terms of the cosets of linear error-correction codes for these virtual channels, a fact first illustrated in [13] for the special case of $m = 2$ terminals connected by a virtual binary symmetric channel (BSC). This fact, exploited by most known linear constructions of SW codes (cf. e.g. [2], [7], [8], [11]), can enable us to translate these constructions and other significant recent developments in capacity-achieving linear codes into new SK and PK constructions. (See also recent independent work [10] for related existence results, as also [12].)

Motivated by these considerations, we seek to devise new *constructive schemes* for secrecy generation. The main technical contribution of this work is the following: we consider four simple models of secrecy generation and show how a new class of secret and private keys can be constructed, based on the SW data compression code from [13]. While we do not specify exactly the linear capacity-achieving channel codes used in the SW step of the procedure, these can be chosen – for instance – from the class of LDPC [8] and turbo codes [7] that have attracted wide attention.

II. PRELIMINARIES

Consider a DMMS with $m \geq 2$ components, with corresponding generic rvs X_1, \dots, X_m taking values in finite alphabets $\mathcal{X}_1, \dots, \mathcal{X}_m$, respectively. Let $\mathbf{X}_i = (X_{i,1}, \dots, X_{i,n})$, $i \in \mathcal{M} = \{1, \dots, m\}$, be n i.i.d. repetitions of rv X_i . Terminals $1, \dots, m$, with respective observations $\mathbf{X}_1, \dots, \mathbf{X}_m$, represent the m users who wish to generate a SK by public communication. These terminals can communicate with each other through broadcasts over a noiseless public channel, possibly interactively in many rounds. In general, a transmission from

a terminal is allowed to be any function of its observations, and of all previous transmissions. Let \mathbf{F} denote collectively all the public transmissions.

Given $\varepsilon > 0$, the rv K_S represents an ε -secret key (ε -SK) for the terminals in \mathcal{M} , achieved with communication \mathbf{F} , if there exist rvs $K_i = K_i(\mathbf{X}_i, \mathbf{F})$, $i \in \mathcal{M}$, with K_i and K_S taking values in the same finite set \mathcal{K}_S such that K_S satisfies

- the common randomness condition

$$\Pr(K_i = K_S, i \in \mathcal{M}) \geq 1 - \varepsilon;$$

- the secrecy condition

$$\frac{1}{n}I(K_S \wedge \mathbf{F}) \leq \varepsilon;$$

- the uniformity condition

$$\frac{1}{n}H(K_S) \geq \frac{1}{n} \log |\mathcal{K}_S| - \varepsilon.$$

Let $A \subset \mathcal{M}$ be an arbitrary subset of terminals. The rv $K_{\mathcal{P}}(A)$ represents an ε -private key (ε -PK) for the terminals in A , private from the terminals in $A^c = \mathcal{M} \setminus A$, achieved with communication \mathbf{F} , if there exist rvs $K_i = K_i(\mathbf{X}_i, \mathbf{F})$, $i \in A$, with K_i and $K_{\mathcal{P}}(A)$ taking values in the same finite set $\mathcal{K}_{\mathcal{P}}(A)$ such that $K_{\mathcal{P}}(A)$ satisfies

- the common randomness condition

$$\Pr(K_i = K_{\mathcal{P}}(A), i \in A) \geq 1 - \varepsilon;$$

- the secrecy condition

$$\frac{1}{n}I(K_{\mathcal{P}}(A) \wedge \{\mathbf{X}_i, i \in A^c\}, \mathbf{F}) \leq \varepsilon;$$

- the uniformity condition

$$\frac{1}{n}H(K_{\mathcal{P}}(A)) \geq \frac{1}{n} \log |\mathcal{K}_{\mathcal{P}}(A)| - \varepsilon.$$

Definition 1 [5]: A nonnegative number R is called an *achievable SK rate* if an ε_n -SK $K_S^{(n)}$ is achievable with suitable communication (with the number of rounds possibly depending on n), such that $\varepsilon_n \rightarrow 0$ and $\frac{1}{n}H(K_S^{(n)}) \rightarrow R$. The largest achievable SK rate is called the *SK-capacity*, denoted by C_{SK} . The PK-capacity for the terminals in A , denoted by $C_{PK}(A)$, is similarly defined. An achievable SK rate (resp. PK rate) will be called *strongly achievable* if ε_n above can be taken to vanish exponentially in n . The corresponding capacities are termed *strong capacities*.

Single-letter characterizations have been provided for C_{SK} in the case of $m = 2$ terminals in [9], [1] and for $m \geq 2$ in [5]; and for $C_{PK}(A)$ in case of $m = 3$ in [1] and for $m \geq 3$ in [5]. The proofs of the achievability parts exploit the close connection between secrecy generation and SW data compression. For instance, “common randomness,” without any secrecy restrictions, is first generated through SW-compressed interterminal communication. This means that all the m terminals acquire a rv with probability $\cong 1$. In the next step, secrecy is then extracted from this common randomness by means of a suitable *identical* operation performed at each terminal on the acquired common randomness. When the

common randomness first acquired by the m terminals is maximal, then the corresponding secret key has the best rate C_{SK} given by (1).

In this work, we consider four simple models for which we illustrate the *construction* of appropriate *strong* secret or private keys, which rely on suitable SW codes. The SW codes of interest will rely on the following result concerning the existence of “good” linear channel codes for a BSC.

Hereafter, a BSC with crossover probability p , $0 < p < \frac{1}{2}$, will be denoted by $\text{BSC}(p)$. Let $h_b(p)$ be the binary entropy function.

Lemma 1 [6]: For each $\varepsilon > 0$, $0 < p < \frac{1}{2}$, and for all n sufficiently large, there exists a binary linear $(n, n - m)$ code for the BSC(p), where $m < n[h_b(p) + \varepsilon]$, such that the average error probability of maximum likelihood decoding is less than $2^{-n\eta}$, for some $\eta > 0$.

III. MAIN RESULTS

MODEL 1: Let the terminals 1 and 2 observe, respectively, n i.i.d. repetitions of the correlated rvs X_1 and X_2 , where X_1, X_2 are $\{0, 1\}$ -valued rvs with joint probability mass function (pmf)

$$P_{X_1 X_2}(x_1, x_2) = \frac{1}{2}(1 - p)\delta_{x_1 x_2} + \frac{1}{2}p(1 - \delta_{x_1 x_2}), \quad p < \frac{1}{2}, \quad (3)$$

with δ being the Kronecker delta function. These two terminals wish to generate a strong SK of maximal rate.

The SK-capacity for this model is [9], [1], [5]

$$C_{SK} = I(X_1 \wedge X_2) = 1 - h_b(p) \text{ bit/symbol.}$$

In the following, we show a simple scheme for both terminals to generate a SK with rate close to $1 - h_b(p)$, which relies on Wyner’s well-known method for SW data compression [13]. The SW problem of interest entails terminal 2 reconstructing the observed sequence \mathbf{x}_1 at terminal 1 from the SW codeword for \mathbf{x}_1 and its own observed sequence \mathbf{x}_2 .

(i) *SW data compression* [13]: Let \mathcal{C} be the linear $(n, n - m)$ code specified in Lemma 1 with parity check matrix \mathbf{P} . Both terminals know \mathcal{C} and \mathbf{P} .

Terminal 1 transmits the syndrome $\mathbf{P}\mathbf{x}_1^t$ to terminal 2. The maximum likelihood estimate of \mathbf{x}_1 at terminal 2 is:

$$\hat{\mathbf{x}}_2(1) = \mathbf{x}_2 \oplus f_{\mathbf{P}}(\mathbf{P}\mathbf{x}_1^t \oplus \mathbf{P}\mathbf{x}_2^t),$$

where $f_{\mathbf{P}}(\mathbf{P}\mathbf{x}_1^t \oplus \mathbf{P}\mathbf{x}_2^t)$ is the most likely n -sequence \mathbf{v} with syndrome $\mathbf{P}\mathbf{v}^t = \mathbf{P}\mathbf{x}_1^t \oplus \mathbf{P}\mathbf{x}_2^t$, with \oplus denoting addition modulo 2 and t denoting transposition.

The probability of decoding error at terminal 2 is given by

$$\Pr(\hat{\mathbf{X}}_2(1) \neq \mathbf{X}_1) = \Pr(\mathbf{X}_2 \oplus f_{\mathbf{P}}(\mathbf{P}\mathbf{X}_1^t \oplus \mathbf{P}\mathbf{X}_2^t) \neq \mathbf{X}_1).$$

Under the given joint pmf (3), \mathbf{X}_2 can be considered as an input to a virtual BSC(p), while \mathbf{X}_1 is the corresponding output, i.e., we can write

$$\mathbf{X}_1 = \mathbf{X}_2 \oplus \mathbf{V},$$

where $\mathbf{V} = (V_1, \dots, V_n)$ is an i.i.d. sequence of $\{0, 1\}$ -valued rvs, independent of \mathbf{X}_2 , with $\Pr(V_i = 1) = p$, $1 \leq i \leq n$. It readily follows that

$$\Pr(\hat{\mathbf{X}}_2(1) \neq \mathbf{X}_1) = \Pr(f_{\mathbf{P}}(\mathbf{P}\mathbf{V}^t) \neq \mathbf{V}).$$

Therefore, it follows from Lemma 1 that for some $\eta > 0$,

$$\Pr(\hat{\mathbf{X}}_2(1) \neq \mathbf{X}_1) < 2^{-n\eta},$$

for all n sufficiently large.

(ii) *SK construction*: Consider a (common) standard array for \mathcal{C} known to both terminals. Denote by $\mathbf{a}_{i,j}$ the element of the i^{th} row and the j^{th} column in the standard array, $1 \leq i \leq 2^m$, $1 \leq j \leq 2^{n-m}$.

Terminal 1 sets $K_1 = j_1$ if \mathbf{X}_1 equals \mathbf{a}_{i,j_1} in the standard array. Terminal 2 sets $K_2 = j_2$ if $\hat{\mathbf{X}}_2(1)$ equals \mathbf{a}_{i,j_2} in the same standard array.

(iii) *SK criteria*: The following theorem shows that K_1 constitutes a strongly achievable SK with rate approaching the SK-capacity.

Theorem 1: The pair of rvs (K_1, K_2) generated above, with (common) range \mathcal{K}_1 (say), satisfy

$$\Pr(K_1 \neq K_2) < 2^{-n\eta};$$

$$I(K_1 \wedge \mathbf{F}) = 0;$$

$$H(K_1) = \log |\mathcal{K}_1|.$$

Further,

$$\frac{1}{n}H(K_1) > 1 - h_b(p) - \varepsilon.$$

Remark: The probability of K_1 being different from K_2 exactly equals the average error probability of maximum likelihood decoding when \mathcal{C} is used on a BSC(p). Furthermore, the gap between the rate of the generated SK and the SK-capacity is as wide as the gap between the rate of \mathcal{C} and the channel capacity. Therefore, if a “better” channel code for a BSC(p), in the sense that the rate of this code is closer to the channel capacity and the average error probability of maximum likelihood decoding is smaller, is applied, then a “better” SK can be generated at both terminals, in the sense that the rate of this SK is closer to the SK-capacity and the probability is smaller that the keys generated at different terminals do not agree with each other.

MODEL 2: Let the terminals 1 and 2 observe, respectively, n i.i.d. repetitions of the correlated rvs X_1 and X_2 , where X_1, X_2 are $\{0, 1\}$ -valued rvs with joint pmf

$$P_{X_1 X_2}(0, 0) = (1 - p)(1 - q),$$

$$P_{X_1 X_2}(0, 1) = pq,$$

$$P_{X_1 X_2}(1, 0) = p(1 - q),$$

$$P_{X_1 X_2}(1, 1) = q(1 - p),$$

where $p < \frac{1}{2}$ and $0 < q < 1$. These two terminals wish to generate a strong SK of maximal rate.

Note that Model 1 is a special case of Model 2 for $q = \frac{1}{2}$. We show below a scheme for both terminals to generate a SK

with rate close to the SK-capacity for this model [9], [1], [5], which is

$$C_{SK} = I(X_1 \wedge X_2) = h_b(p + q - 2pq) - h_b(p) \text{ bit/symbol.}$$

(i) *SW data compression*: This step is identical to step (i) for Model 1.

(ii) *SK construction*: Suppose that both terminals know the linear $(n, n - m)$ code \mathcal{C} specified in Lemma 1, and a (common) standard array for \mathcal{C} . Let $\{e_i : 1 \leq i \leq 2^m\}$ denote the set of coset leaders for all the cosets of \mathcal{C} . Given a (generic) $\{0, 1\}$ -valued rv X , the set of sequences $\mathbf{x} \in \{0, 1\}^n$ is called *X-typical with constant ξ* , denoted by $T_{X, \xi}^n$, if

$$2^{-n[H(X)+\xi]} \leq P_{X, \xi}^n(\mathbf{x}) \leq 2^{-n[H(X)-\xi]}.$$

Denote by A_i the set of $T_{X, \xi}^n$ sequences in the coset of \mathcal{C} with coset leader e_i , $1 \leq i \leq 2^m$. If the number of sequences of the same type (cf. [4]) in A_i is more than $2^{n[I(X_1 \wedge X_2) - \varepsilon']}$, where $\varepsilon' > \xi + \varepsilon$, then collect arbitrarily $2^{n[I(X_1 \wedge X_2) - \varepsilon']}$ such sequences to compose a subset, which we call a *regular subset* (as it consists of sequences of the same type). Continue this procedure until the number of sequences of every type in A_i is less than $2^{n[I(X_1 \wedge X_2) - \varepsilon']}$. Let N_i denote the number of distinct regular subsets of A_i .

Enumerate (in any way) the sequences in each regular subset. Let $\mathbf{b}_{i,j,k}$, where $1 \leq i \leq 2^m$, $1 \leq j \leq N_i$, $1 \leq k \leq 2^{n[I(X \wedge Y) - \varepsilon']}$, denote the k^{th} sequence of the j^{th} regular subset in the i^{th} coset (i.e., the coset with coset leader e_i).

Terminal 1 sets $K_1 = k_1$ if \mathbf{X}_1 equals \mathbf{b}_{i,j_1,k_1} . Otherwise, K_1 is set to be uniformly distributed on $\{1, \dots, 2^{n[I(X_1 \wedge X_2) - \varepsilon']}\}$, and independent of $(\mathbf{X}_1, \mathbf{X}_2)$. Terminal 2 sets $K_2 = k_2$ if $\hat{\mathbf{X}}_2(1)$ equals \mathbf{b}_{i,j_2,k_2} . Otherwise, K_2 is set to be uniformly distributed on $\{1, \dots, 2^{n[I(X_1 \wedge X_2) - \varepsilon']}\}$, independent of $(\mathbf{X}_1, \mathbf{X}_2, K_1)$.

(iii) *SK criteria*: The following theorem shows that K_1 constitutes a strongly achievable SK with rate approaching the SK-capacity.

Theorem 2: For some $\eta' = \eta'(\eta, \xi, \varepsilon, \varepsilon') > 0$, the pair of rvs (K_1, K_2) generated above, with range \mathcal{K}_1 (say), satisfy

$$\Pr(K_1 \neq K_2) < 2^{-n\eta'};$$

$$I(K_1 \wedge \mathbf{F}) = 0;$$

$$H(K_1) = \log |\mathcal{K}_1|.$$

Further,

$$\frac{1}{n} H(K_1) = I(X_1 \wedge X_2) - \varepsilon'.$$

MODEL 3: Let the terminals $1, \dots, m$ observe, respectively, n i.i.d. repetitions of $\{0, 1\}$ -valued rvs X_1, \dots, X_m which form a Markov chain

$$X_1 \text{ --- } X_2 \text{ --- } \dots \text{ --- } X_m,$$

with a joint pmf $P_{X_1 \dots X_m}$ given by: for $1 \leq i \leq m - 1$,

$$P_{X_i X_{i+1}}(x_i, x_{i+1}) = \frac{1}{2}(1-p_i)\delta_{x_i x_{i+1}} + \frac{1}{2}p_i(1-\delta_{x_i x_{i+1}}), \quad p_i < \frac{1}{2}.$$

These m terminals wish to generate a strong SK of maximal rate.

Note that Model 1 is a special case of Model 3 for $m = 2$. Without any loss of generality, let

$$p_j = \max_{1 \leq i \leq m-1} p_i.$$

Then, the SK-capacity for this model is [5]

$$C_{SK} = I(X_j \wedge X_{j+1}) = 1 - h_b(p_j) \text{ bit/symbol.}$$

We show below how to extract a SK with rate close to $1 - h_b(p_j)$ by using a SW data compression scheme for reconstructing \mathbf{x}_j at all the terminals.

(i) *SW data compression*: Let \mathcal{C} be the linear $(n, n - m)$ code specified in Lemma 1 for the BSC(p_j), with parity check matrix \mathbf{P} . Terminals i , $1 \leq i \leq m - 1$, transmit the syndromes $\mathbf{P}\mathbf{x}_i^t$, respectively.

Let $\hat{\mathbf{x}}_i(j)$ denote the maximum likelihood estimate at terminal i of \mathbf{x}_j . For $1 \leq i \leq j - 1$, terminal i , with the knowledge of $(\mathbf{P}\mathbf{x}_{i+1}^t, \dots, \mathbf{P}\mathbf{x}_j^t, \mathbf{x}_i)$, forms the following successive maximum likelihood estimates

$$\begin{aligned} \hat{\mathbf{x}}_i(i+1) &= \mathbf{x}_i \oplus f_{\mathbf{P}}(\mathbf{P}\mathbf{x}_i^t \oplus \mathbf{P}\mathbf{x}_{i+1}^t), \\ \hat{\mathbf{x}}_i(i+2) &= \hat{\mathbf{x}}_i(i+1) \oplus f_{\mathbf{P}}(\mathbf{P}\mathbf{x}_{i+1}^t \oplus \mathbf{P}\mathbf{x}_{i+2}^t), \\ &\vdots \\ \hat{\mathbf{x}}_i(j) &= \hat{\mathbf{x}}_i(j-1) \oplus f_{\mathbf{P}}(\mathbf{P}\mathbf{x}_{j-1}^t \oplus \mathbf{P}\mathbf{x}_j^t). \end{aligned}$$

For $j+1 \leq i \leq m$, terminal i , with the knowledge of $(\mathbf{P}\mathbf{x}_j^t, \dots, \mathbf{P}\mathbf{x}_{i-1}^t, \mathbf{x}_i)$, forms the following successive maximum likelihood estimates

$$\begin{aligned} \hat{\mathbf{x}}_i(i-1) &= \mathbf{x}_i \oplus f_{\mathbf{P}}(\mathbf{P}\mathbf{x}_i^t \oplus \mathbf{P}\mathbf{x}_{i-1}^t), \\ \hat{\mathbf{x}}_i(i-2) &= \hat{\mathbf{x}}_i(i-1) \oplus f_{\mathbf{P}}(\mathbf{P}\mathbf{x}_{i-1}^t \oplus \mathbf{P}\mathbf{x}_{i-2}^t), \\ &\vdots \\ \hat{\mathbf{x}}_i(j) &= \hat{\mathbf{x}}_i(j+1) \oplus f_{\mathbf{P}}(\mathbf{P}\mathbf{x}_{j+1}^t \oplus \mathbf{P}\mathbf{x}_j^t). \end{aligned}$$

It can be shown that for some $\eta' = \eta'(\eta, m) > 0$,

$$\Pr(\hat{\mathbf{X}}_i(j) = \mathbf{X}_j, 1 \leq i \neq j \leq m) > 1 - 2^{-n\eta'}.$$

(ii) *SK construction*: Consider a (common) standard array for \mathcal{C} known to all the terminals. Denote by $\mathbf{a}_{l,k}$ the element of the l^{th} row and the k^{th} column in the standard array, $1 \leq l \leq 2^m$, $1 \leq k \leq 2^{n-m}$.

Terminal j sets $K_j = k_j$ if \mathbf{X}_j equals \mathbf{a}_{l,k_j} in the standard array. Terminal i , $1 \leq i \neq j \leq m$, sets $K_i = k_i$ if $\hat{\mathbf{X}}_i(j)$ equals \mathbf{a}_{l,k_i} in the same standard array.

(iii) *SK criteria*: The following theorem shows that K_j constitutes a strongly achievable SK with rate approaching the SK-capacity.

Theorem 3: The set of rvs (K_1, \dots, K_m) generated above, with range \mathcal{K}_j (say), satisfy

$$\Pr(K_1 = \dots = K_m) > 1 - 2^{-n\eta'};$$

$$I(K_j \wedge \mathbf{F}) = 0;$$

$$H(K_j) = \log |\mathcal{K}_j|.$$

Further,

$$\frac{1}{n}H(K_j) > 1 - h_b(p_j) - \varepsilon.$$

MODEL 4: Let the terminals 1, 2 and 3 observe, respectively, n i.i.d. repetitions of the correlated rvs X_1, X_2, X_3 , where X_1, X_2, X_3 are $\{0, 1\}$ -valued rvs with joint pmf

$$\begin{aligned} P_{X_1 X_2 X_3}(0, 0, 0) &= P_{X_1 X_2 X_3}(0, 1, 1) = \frac{(1-p)(1-q)}{2}, \\ P_{X_1 X_2 X_3}(0, 0, 1) &= P_{X_1 X_2 X_3}(0, 1, 0) = \frac{pq}{2}, \\ P_{X_1 X_2 X_3}(1, 0, 0) &= P_{X_1 X_2 X_3}(1, 1, 1) = \frac{p(1-q)}{2}, \\ P_{X_1 X_2 X_3}(1, 0, 1) &= P_{X_1 X_2 X_3}(1, 1, 0) = \frac{q(1-p)}{2}, \end{aligned}$$

where $p < \frac{1}{2}$ and $0 < q < 1$. Terminals 1 and 2 wish to generate a strong PK of maximal rate, which is concealed from the helper terminal 3.

Note that under the given joint pmf of X_1, X_2, X_3 , we can write

$$\mathbf{X}_1 = \mathbf{X}_2 \oplus \mathbf{X}_3 \oplus \mathbf{V},$$

where $\mathbf{V} = (V_1, \dots, V_n)$ is an i.i.d. sequence of $\{0, 1\}$ -valued rvs, independent of $(\mathbf{X}_2, \mathbf{X}_3)$, with $\Pr(V_i = 1) = p, 1 \leq i \leq n$.

We show below a scheme for terminals 1 and 2 to generate a PK with rate close to the PK-capacity for this model [1], [5]

$$\begin{aligned} C_{PK}(\{1, 2\}) &= I(X_1 \wedge X_2 | X_3) \\ &= h_b(p + q - 2pq) - h_b(p) \text{ bit/symbol.} \end{aligned}$$

The first step of this scheme entails terminal 3 simply revealing its observations \mathbf{x}_3 to both terminals 1 and 2. Then, Wyner's SW data compression scheme is used for reconstructing \mathbf{x}_1 at terminal 2 from the SW codeword for \mathbf{x}_1 and $\mathbf{x}_2 \oplus \mathbf{x}_3$.

(i) *SW data compression:* This step is identical to step (i) for Model 1.

(ii) *SK construction:* Suppose that terminals 1 and 2 know the linear $(n, n - m)$ code \mathcal{C} specified in Lemma 1, and a (common) standard array for \mathcal{C} . Let $\{\mathbf{e}_i : 1 \leq i \leq 2^m\}$ denote the set of coset leaders for all the cosets of \mathcal{C} . Given (generic) $\{0, 1\}$ -valued rvs X, Y , the set of pairs of sequences $(\mathbf{x}, \mathbf{y}) \in \{0, 1\}^n \times \{0, 1\}^n$ is called *XY -typical with constant ξ* , denoted by $T_{XY, \xi}^n$, if $\mathbf{x} \in T_{X, \xi}^n, \mathbf{y} \in T_{Y, \xi}^n$, and

$$2^{-n[H(X, Y) + \xi]} \leq P_{XY}^n(\mathbf{x}, \mathbf{y}) \leq 2^{-n[H(X, Y) - \xi]}.$$

For every $\mathbf{y} \in \{0, 1\}^n$, the set of sequences $\mathbf{x} \in \{0, 1\}^n$ is called *$X|Y$ -typical with respect to \mathbf{y} with constant ξ* , denoted by $T_{X|Y, \xi}^n(\mathbf{y})$, if $(\mathbf{x}, \mathbf{y}) \in T_{XY, \xi}^n$. Note that $T_{X|Y, \xi}^n(\mathbf{y})$ is an empty set if $\mathbf{y} \notin T_{Y, \xi}^n$.

For a sequence $\mathbf{x}_3 \in \{0, 1\}^n$, denote by $A_i(\mathbf{x}_3)$ the set of $T_{X_1|X_3, \xi}^n(\mathbf{x}_3)$ sequences in the coset of \mathcal{C} with coset leader $\mathbf{e}_i, 1 \leq i \leq 2^m$. If the number of sequences of the same joint type (cf. [4]) with \mathbf{x}_3 in $A_i(\mathbf{x}_3)$ is more than $2^{n[I(X_1 \wedge X_2 | X_3) - \varepsilon']}$, where $\varepsilon' > 2\xi + \varepsilon$, then collect arbitrarily $2^{n[I(X_1 \wedge X_2 | X_3) - \varepsilon']}$ such sequences to compose a regular subset.

Continue this procedure until the number of sequences of every joint type with \mathbf{x}_3 in $A_i(\mathbf{x}_3)$ is less than $2^{n[I(X_1 \wedge X_2 | X_3) - \varepsilon']}$. Let $N_i(\mathbf{x}_3)$ denote the number of distinct regular subsets of $A_i(\mathbf{x}_3)$.

For a given sequence \mathbf{x}_3 , enumerate (in any way) the sequences in each regular subset. Let $\mathbf{b}_{i,j,k}(\mathbf{x}_3)$, where $1 \leq i \leq 2^m, 1 \leq j \leq N_i(\mathbf{x}_3), 1 \leq k \leq 2^{n[I(X_1 \wedge X_2 | X_3) - \varepsilon']}$, denote the k^{th} sequence of the j^{th} regular subset in the i^{th} coset.

Terminal 1 sets $K_1 = k_1$ if \mathbf{X}_1 equals $\mathbf{b}_{i,j_1,k_1}(\mathbf{X}_3)$. Otherwise, K_1 is set to be uniformly distributed on $\{1, \dots, 2^{n[I(X_1 \wedge X_2 | X_3) - \varepsilon]}\}$, independent of $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$. Terminal 2 sets $K_2 = k_2$ if $\mathbf{X}_2(1)$ equals $\mathbf{b}_{i,j_2,k_2}(\mathbf{X}_3)$. Otherwise, K_2 is set to be uniformly distributed on $\{1, \dots, 2^{n[I(X_1 \wedge X_2 | X_3) - \varepsilon]}\}$, independent of $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, K_1)$.

(iii) *SK criteria:* The following theorem shows that K_1 constitutes a strongly achievable PK with rate approaching the PK-capacity.

Theorem 4: For some $\eta' = \eta'(\eta, \xi, \varepsilon, \varepsilon') > 0$, the pair of rvs (K_1, K_2) generated above, with range \mathcal{K}_1 (say), satisfy

$$\Pr(K_1 \neq K_2) < 2^{-n\eta'};$$

$$I(K_1 \wedge \mathbf{X}_3, \mathbf{F}) = 0;$$

$$H(K_1) = \log |\mathcal{K}_1|.$$

Further,

$$\frac{1}{n}H(K_1) = I(X_1 \wedge X_2 | X_3) - \varepsilon'.$$

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