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JOINTLY OPTIMAL ADMISSION AND ROUTING CONTROLS AT A NETWORK NODE

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ABSTRACT

We consider the problem of jointly optimal admission and routing at a data network node. Specifically, a message arriving at the buffer of a node in a data network is to be transmitted over one of two channels with different transmission times. Under suitably chosen criteria, two decisions have to be made: Whether or not to admit an incoming message into the buffer, and under what conditions should the slower channel be utilized. A discounted infinite-horizon cost as well as an average cost are considered. These costs consist of a linear combination of the blocking probability and the queueing delay at the buffer. The optimal admission and routing strategies are shown to be characterized almost completely by means of "switching curves."

1. Introduction

Admission control and routing are key issues arising in the design and operation of communication and computer networks, and have received considerable attention in recent years. The admission control problem entails a determination of efficient policies for allowing incoming messages to gain access to network facilities. The routing problem involves selecting paths from several alternatives in the network along which accepted messages can be efficiently forwarded to their destinations.

Numerous studies of admission control and routing problems at a single node or at several nodes of a network can be found in the literature. Decisions for allowing messages into a network have customarily been based on an appropriate minimization of a blocking cost in conjunction with a cost for queuing delays in the buffers at the nodes. Routing strategies, on the other hand, have typically been determined using the queuing delays at the buffers as the measure of performance. We cite below some of the studies relevant to our work; this list is by no means exhaustive.

Stidham [24] has considered admission control policies for several simple queuing models. The optimal admission control policies for all these models share the characteristic that they can be expressed in terms of a "switching curve." Viniotis-Ephremides [28] have demonstrated a similar characterization of the optimal admission strategy at a simple node in an Integrated Services Digital Network (ISDN). Results in the same vein have been obtained by Christidou *et al* [1] for a cyclic interconnection of two queues, and by Lambadaris *et al* [9] for a circuit-switched node. Hajek [2] has investigated the problem of optimally controlling two interacting queues.

In the realm of relevant routing problems, Lin-Kumar [10] have considered the task of routing messages arriving at a node among two channels (servers), one faster than the other. By minimizing the average queuing delay at the node buffer, they show that the optimal routing policy is characterized by a "threshold" on the size of the queue. Rosberg-Makowski [16] have treated a similar problem involving multiple servers under the assumption of light traffic. In a recent preprint, Luh -Viniotis [13] claim the optimality of a policy determined by multiple thresholds for the situation in [16] even with arbitrary arrival rates. Nain-Ross [14] consider the optimal assignment of a single server to multiple classes of customers. In doing so, they minimize a linear combination of the average queue lengths of the various classes of customers while simultaneously constraining the average queue length of a specific customer class to lie below a specified value. Shwartz-Makowski [23] treat a similar problem with two types of customers. Both [14,23] show the optimal assignment strategies to be random assignments.

In what follows, we combine the elements of the admission control and routing problems at a simple node of a communication network similar to that studied in Lin-Kumar[10]. To our knowledge, this is the first determination of *simultaneously optimal policies* for flow control and routing. In our model, a message arriving at a buffer is to be transmitted over one of two channels with different transmission times. Under suitably chosen criteria, two decisions have to be made: whether or not to admit an incoming message into the buffer, and under what conditions should the slower channel be utilized. A discounted infinite-horizon cost as well as an average cost are considered which consist of a linear combination of the blocking probability and the queuing delay at the buffer.

Beginning with the discounted cost case, we formulate the optimal control task as a Markov decision problem. It is first shown from Lippman [11,12] that an optimal policy exists for admission and routing which is stationary in nature. Next, properties of the optimal cost function are derived using arguments which rely heavily on modifications of the sample path methods [29] as well as of the linear programming approach developed by Rosberg *al* [17]. The said properties are then used to demonstrate that the optimal admission and routing controls are characterized almost completely by "switching curves." This task is complicated by the appearance of the relevant control terms in a nonlinear fashion in the associated dynamic programming equations; nonetheless, it is possible to perform the necessary minimizations, albeit tediously, to derive the optimal controls. Finally, we show that the average-cost problem also yields similar results; some elaborate arguments are needed which are partly provided by the approach of Sennott [22].

The remainder of this paper is organized as follows. The problem is formulated in section 2. Section 3 considers the discounted-cost case and establishes key properties of the optimal discounted-cost function. The associated optimal policy

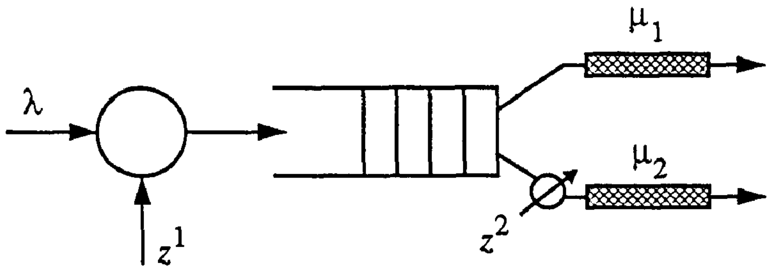

Figure 1: The queueing model

for this case is characterized in section 4; to do so, we need some convexity properties of the optimal discounted-cost which are established in section 5. Finally, the average-cost problem is addressed in section 6.

2. Problem Statement and Preliminaries

The model under consideration is shown in Figure 1. We focus our attention on a single node of a communication network providing service to a stream of message packets that arrive according to a Poisson distribution with parameter λ.

The packets (customers) are stored in a buffer (queue), and subsequently are to be routed through one of channels (servers) 1 or 2 which have transmission times that are exponentially distributed with parameters μ_i, $i = 1, 2$. We assume that transmission over channel 1 is faster than that over channel 2, i.e., $\mu_1 > \mu_2$, and that channel 1 is non-idling. That the faster channel is non-idling in our set-up can is easily be shown in Appendix 1 by sample path arguments *a la* Walrand [29]. Furthermore, in order to ensure that the number of packets in the buffer remains bounded we shall assume the standard stability condition, $\lambda < \mu_1 + \mu_2$.

The objective is the following: We wish to simultaneously control the admission of packets to the buffer, as well as their subsequent allocation to the two channels; this will be done in such a way as to minimize a weighted sum of the probability of rejecting admission of an arriving packet to the buffer and the delay experienced by the packets in the queue. This problem can be precisely formulated in terms of a Markov decision process (MDP) [3,6,20] as follows.

The *state* of the system at time $t, t \geq 0$, is defined by a stochastic process $(\mathbf{x}_t,\ t \geq 0)$, describing the evolution of the total load of the system as well as of the status of the slower channel, where $\mathbf{x}_t = (x_t^1, x_t^2)$ takes values in the *state space* $\mathcal{S} = (0,0) \cup (Z^+ \times \{0,1\})$, with

$$x_t^1 = \text{ total number of packets in the system (including the two channels)},$$

$$x_t^2 = \begin{cases} 0 & \text{if channel 2 is empty of packets} \\ 1 & \text{if channel 2 is servicing a packet,} \end{cases}$$

at time t. $Z^+ \times \{0,1\}$ denotes the cartesian product of Z^+ and $\{0,1\}$. Observe that the process $(\mathbf{x}_t, t \geq 0)$ is piecewise constant; we shall assume that its sample paths are right-continous. Next, we associate with each state $\mathbf{x}$ in $\mathcal{S}$ a set of admissible actions $\mathcal{D} = \{0,1\}^2$. Thus, an admissible action $\mathbf{z}_t(\mathbf{x})$ in state $\mathbf{x}$ at time t, with values in $\mathcal{D}$ will have the form

$$\mathbf{z}_t(\mathbf{x}) = (z_t^1(\mathbf{x}), z_t^2(\mathbf{x}))$$

where $z^1 = 1$ or 0 according to whether an arriving packet is accepted into the buffer or is rejected (and lost), and $z^2 = 1$ or 0 according to whether or not the slower channel 2 is activated.

Defining the *action space* to be the set $\mathcal{A} = \mathcal{D}^{\mathcal{S}}$, we can now represent an *admissible control strategy* (CS) as an $\mathcal{A}$-valued stochastic process $(z_t,\ t \geq 0)$, where $z_t = (\mathbf{z}_t(\mathbf{x}),\ \mathbf{x} \in \mathcal{S})$. Hereafter, we shall use the abbreviated notation z for the CS $(z_t,\ t \geq 0)$. Let $\mathcal{P}$ denote the set of all admissible control strategies.

A *law of motion* corresponding to a CS z is specified by a transition probability $\mathbb{P}(\mathbf{x}'|\mathbf{x}, z_t),\ \mathbf{x}, \mathbf{x}' \in \mathcal{S},\ t \geq 0$, denoting the conditional probability that the system moves to state $\mathbf{x}\prime$ at time t^+ when the action $\mathbf{z}_t(\mathbf{x})$ is applied to it at time t while in state $\mathbf{x}$.

Our objective is to find a CS z in $\mathcal{P}$ minimizing the following cost:

$$\limsup_{T \to \infty} \mathbb{E}_{\mathbf{x}}^{z} \left(\frac{1}{T} \int_0^T (1 - z_t^1(\mathbf{x}_t) + \gamma x_t^1) dt \right) \qquad \gamma > 0, \qquad (P1)$$

where $\mathbb{E}_{\mathbf{x}}^{z}$ denotes expectation with respect to the probability measure induced by the CS z on the process $(\mathbf{x}_t,\ t \geq 0)$ with initial state $\mathbf{x}$ at $t = 0$. If such a minimizing CS exists, we shall refer to it as the optimal strategy for the *unconstrained average cost problem* (P1).

A key step in treating (P1) involves the discounted cost problem associated with (P1). Namely we wish to find a CS z in $\mathcal{P}$ for which the following discounted cost [3,20] is minimized:

$$\limsup_{T\to\infty} \mathbb{E}_{\mathbf{x}}^{z} \left(\int_0^T e^{-\delta t} \left(1 - z_t^1(\mathbf{x}_t) + \gamma x_t^1\right) dt \right) \qquad \delta > 0, \quad \gamma > 0. \qquad (P2)$$

If such a minimizing CS exists, it is called the optimal strategy for the *discounted cost* problem (P2).

We conclude this section by introducing two special classes of relevant CS's. An admissible CS which is an i.i.d. stochastic process will be called a *stationary randomized strategy (SRS).* Furthermore, if the common distribution of the SRS z has all its mass concentrated at some point in $\mathcal{A}$, we shall refer to it as a *stationary strategy (SS).* Let $\mathcal{P}_S \subset \mathcal{P}$ denote the set of all SS's.

Problems $P1$ and $P2$ are closely related. Considering first the discounted cost problem ($P2$), we assert from Lippman [11, p.1238] that an optimal CS exists which, furthermore, is stationary. To this end, first observe that the cost incurred in state $\mathbf{x}_t = (x_t^1, x_t^2)$ at time t has at most a linear growth with respect to x^1, i.e.,

$$1 - z_t^1(\mathbf{x}_t) + \gamma x_t^1 \leq 1 + \gamma x_t^1.$$

Next, the inter-arrival and inter-departure times of the packets are exponentially distributed. Furthermore, the action set $\mathcal{D}$ is finite. The assumptions of [11, Thm. 1 p. 1239] are thereby satisfied, leading to our assertion above.

Hereafter, we replace $\mathbf{z}_t(\mathbf{x}_t)$ by $\mathbf{z}_t$ for notational convenience. Furthermore, in view of our previous assertion, we restrict attention to stationary CS's and define the δ-discounted cost starting with initial state $\mathbf{x}$ associated with the problem (P2) by

$$J^{\gamma,\delta}(\mathbf{x}) \triangleq \min_{z\in\mathcal{P}_S} \mathbb{E}_{\mathbf{x}}^{z} \left(\int_0^\infty e^{-\delta t}(1 - z_t^1 + \gamma x_t^1) dt \right) \qquad \delta > 0, \quad \gamma > 0. \qquad (2.1)$$

The minimum cost in (2.1) can be expressed in an alternative form which facilitates further analysis. To this end let $0 = t_0 < t_1 < t_2 < \cdots < t_n \cdots$ be the (random) instants in time denoting transition epochs of the system state $(\mathbf{x}_t,\ t \geq 0)$, where each transition epoch represents either an arrival of a packet into, or a departure of a packet from, the system. It is convenient to introduce at this point the δ-discounted expected cost over the time-horizon $[0, t_n)$, with initial state $\mathbf{x}$, and following a control strategy z in $\mathcal{P}_S$, namely,

$$V_n^{\gamma,\delta}(\mathbf{x}, z) \triangleq \mathbb{E}_{\mathbf{x}}^z \left(\int_0^{t_n} e^{-\delta t}(1 - z_t^1 + \gamma x_t^1) dt \right). \tag{2.2}$$

Let

$$J_n^{\gamma,\delta}(\mathbf{x}) = \min_{z \in \mathcal{P}_S} V_n^{\gamma,\delta}(\mathbf{x}, z), \quad n = 0, 1, \cdots,$$

$$J_\infty^{\gamma,\delta}(\mathbf{x}) = \lim_{n \to \infty} J_n^{\gamma,\delta}(\mathbf{x}).$$

We show in appendix 2 that the minimum cost in (2.1) has the alternative expression

$$J^{\gamma,\delta}(\mathbf{x}) = J_\infty^{\gamma,\delta}(\mathbf{x}) \tag{2.3}$$

for every initial state $\mathbf{x}$. The assertion in (2.3) is standard [cf. e.g., 6] for problems involving finite state spaces. Our problem involves an infinite state space, thereby requiring additional arguments from [11] which are provided in appendix 2.

3. Properties of the Optimal Discounted Cost Functions

We now derive a few properties of the cost functions $J_n^{\gamma,\delta}(\cdot)$ and $J_\infty^{\gamma,\delta}(\cdot)$ which will be employed in the next section to characterize the optimal policy for the discounted cost problem (P2). In order to avoid repetition, the notation $J_*^{\gamma,\delta}(\cdot)$ will be used to represent $J_n^{\gamma,\delta}(\cdot)$, $n = 0, 1, \cdots$, as well as $J_\infty^{\gamma,\delta}(\cdot)$, as appropriate. Note that these properties are valid for every $\gamma > 0$, $\delta > 0$.

Proposition 1: For each x^2, $J_*^{\gamma,\delta}(\cdot, x^2)$ is nondecreasing.

Proof: We shall first prove that $J_n^{\gamma,\delta}(x^1 + 1, x^2) \geq J_n^{\gamma,\delta}(x^1, x^2)$ for all x^1 using the following coupling argument. Consider two similar systems starting with initial conditions (x^1, x^2) and $(x^1 + 1, x^2)$, respectively. Couple the arrival and service processes of both systems; further, in both cases, follow the optimal CS for the latter system starting with initial state $(x^1 + 1, x^2)$. Denoting this strategy by

$z = (z^1, z^2)$, let $(\tilde{\mathbf{x}}_t)$ and $(\mathbf{x}_t)$, $0 \le t < t_n$, be the corresponding trajectories of the systems starting at (x^1, x^2) and $(x^1 + 1, x^2)$, respectively. Define a stopping time by $\tau = [\min(t : \mathbf{x}_t = \tilde{\mathbf{x}}_t)] \wedge t_n$. We see that

$$\int_0^{t_n} e^{-\delta t}(1 - z_t^1 + \gamma x_t^1)dt - \int_0^{t_n} e^{-\delta t}(1 - z_t^1 + \gamma \tilde{x}_t^1)dt \ge \int_0^{\tau} \gamma e^{-\delta t} dt \ge 0$$

so that

$$\begin{aligned} J_n^{\gamma,\delta}(x^1 + 1, x^2) &= \mathbb{E}^z_{(x^1+1,x^2)} \int_0^{t_n} e^{-\delta t}(1 - z_t^1 + \gamma x_t^1)dt \\ &\ge \mathbb{E}^z_{(x^1,x^2)} \int_0^{t_n} e^{-\delta t}(1 - z_t^1 + \gamma \tilde{x}_t^1)dt \\ &\ge J_n^{\gamma,\delta}(x^1, x^2). \end{aligned}$$

The proof of the claim is completed by letting $n \to \infty$, when it readily follows that $J_\infty^{\gamma,\delta}(x^1 + 1, x^2) \ge J_\infty^{\gamma,\delta}(x^1, x^2)$.

Propositions 2-4 and Corollaries 2,3 below stem from arguments very similar to those in [29]. Brief proofs for the interested user are provided in appendix 3.

Proposition 2: For every $\delta > 0$, there exists an integer $\overline{x} = \overline{x}(\delta)$ such that:

$$J^{\gamma,\delta}(\overline{x}, 0) \ge J^{\gamma,\delta}(\overline{x}, 1). \tag{3.1}$$

Corollary 2: There exists a strictly increasing sequence of positive integers $\{x_k\}_{k=1}^{\infty}$ satisfying

$$J^{\gamma,\delta}(x_k, 0) \ge J^{\gamma,\delta}(x_k, 1)$$

for $k \ge 1$.

Corollary 3: If the integers x_1 and x_2, $x_1 < x_2$, satisfy

$$J^{\gamma,\delta}(x_k, 0) \ge J^{\gamma,\delta}(x_k, 1), \quad k = 1, 2,$$

then for all $x_1 < x < x_2$, it holds that

$$J^{\gamma,\delta}(x, 0) \ge J^{\gamma,\delta}(x, 1).$$

Proposition 3: For each x, $J_*^{\gamma,\delta}(x, 0) \le J_*^{\gamma,\delta}(x + 1, 1)$.

Proposition 4: $J_*^{\gamma,\delta}(1, 1) \ge J_*^{\gamma,\delta}(1, 0)$.

Finally, we introduce two more propositions. Their proofs are more involved than those of the previous propositions, and will be provided in section 5 below.

Proposition 5_(Convexity):_ For each $n \geq 0$, $\delta > 0$, and $x^2 = 0,1$, $J_*^{\gamma,\delta}(\cdot, x^2)$ is a convex function, i.e.,

$$J_*^{\gamma,\delta}(x^1+1,x^2) - J_*^{\gamma,\delta}(x^1,x^2) \geq J_*^{\gamma,\delta}(x^1,x^2) - J_*^{\gamma,\delta}(x^1-1,x^2) \tag{3.2}$$

for all $x^1 > 1$.

Proposition 6: For each $n \geq 0$, $\delta > 0$,

$$J_*^{\gamma,\delta}(x^1+1,1) - J_*^{\gamma,\delta}(x^1,0) \geq J_*^{\gamma,\delta}(x^1,1) - J_*^{\gamma,\delta}(x^1-1,0) \tag{3.3}$$

for all $x^1 > 1$.

4. An Optimal Policy for the Discounted Cost Problem

In this section we derive the form of the optimal strategy associated with the β-discounted cost problem (P2). This is done below in two steps.

The first step entails converting the original continuous-time problem (P2) into its discrete-time equivalent by the standard procedure of "uniformization" [2,6,17]. We recall from section 3 that $0 = t_0 < t_1 < t_2 < \cdots < t_n \cdots$ are the (random) instants in time denoting transition epochs of the system state. By suitably introducing dummy departures as in [12,17], the inter-epoch intervals are seen to be i.i.d. random variables with distribution

$$Pr[t_{k+1} - t_k > t] = e^{-t(\lambda+\mu_1+\mu_2)} \tag{4.1}$$

for $k = 0,1,\cdots$. Consider the discrete time system obtained as in [6,17] by sampling the original continous-time system at its transition epochs. To this end, we introduce the notation $\mathbf{x}_k \triangleq \mathbf{x}_{t_k}$ and $\mathbf{z}_k \triangleq \mathbf{z}(\mathbf{x}_{t_k})$ and define

$$\beta = \frac{\lambda+\mu_1+\mu_2}{\lambda+\mu_1+\mu_2+\delta}, \tag{4.2}$$

whence $0 < \beta < 1$. The β-discounted cost incurred by the n-step discrete time system for the CS z is defined [6,17] as

$$\tilde{V}_n^{\gamma,\beta}(\mathbf{x}, z) \triangleq \mathbb{E}_{\mathbf{x}}^{z} \sum_{k=0}^{n-1} \beta^k (1 - z_k^1 + \gamma x_k^1).$$

It then follows as in [6,17] that (cf. (3.4))

$$V_n^{\gamma,\delta}(\mathbf{x}, z) = \frac{1-\beta}{\delta} \tilde{V}_n^{\gamma,\beta}(\mathbf{x}, z). \tag{4.3}$$

Let

$$\tilde{V}^{\gamma,\beta}(\mathbf{x}, z) \triangleq \lim_{n\to\infty} \tilde{V}_n^{\gamma,\beta}(\mathbf{x}, z).$$

We can now state the minimization problem (P2) in terms of a discrete-time problem of equivalent cost as follows. Define the minimum β-discounted cost for the n-step and infinite horizon discrete-time systems, respectively, by

$$\tilde{J}_n^{\gamma,\beta}(\mathbf{x}) \triangleq \min_{z\in\mathcal{P}} \tilde{V}_n^{\gamma,\beta}(\mathbf{x}, z) \tag{4.4}$$

and

$$\tilde{J}^{\gamma,\beta}(\mathbf{x}) \triangleq \min_{z\in\mathcal{P}} \tilde{V}^{\gamma,\beta}(\mathbf{x}, z). \tag{4.5}$$

Letting

$$\tilde{J}_\infty^{\gamma,\beta}(\mathbf{x}) \triangleq \lim_{n\to\infty} \tilde{J}_n^{\beta,\gamma}(\mathbf{x}) \tag{4.6}$$

it can be easily deduced, as in section 3, that

$$\tilde{J}_\infty^{\beta,\gamma}(\mathbf{x}) = \tilde{J}^{\beta,\gamma}(\mathbf{x})$$

for every initial condition $\mathbf{x}$. Finally, the equivalence, in the sense of optimal discounted cost, between (P2) and the discrete-time formulation above follows readily from (4.3) and (4.6) by noting that

$$J^{\gamma,\delta}(\mathbf{x}) = \frac{1-\beta}{\delta} \tilde{J}^{\gamma,\beta}(\mathbf{x}). \tag{4.7}$$

Thus, it suffices to restrict attention hereafter to the discrete-time β-discounted cost problem defined by (4.5).

We can now proceed to the second step associated with problem (P2) by developing the dynamic programming equations for the problem in (4.5). The notation is considerably simplified by introducing the following quantities:

$$A_i = \lambda\beta\left(\tilde{J}^{\gamma,\beta}(i+1,0) - \tilde{J}^{\gamma,\beta}(i,0)\right) - 1, \quad i \geq 0$$

$$B_i = \lambda\beta\left(\tilde{J}^{\gamma,\beta}(i,1) - \tilde{J}^{\gamma,\beta}(i,0)\right) + \beta\mu_1\left(\tilde{J}^{\gamma,\beta}(i-1,1) - \tilde{J}^{\gamma\beta}(i-1,0)\right), \quad i \geq 2$$

$$B_1 = \lambda\beta\left(\tilde{J}^{\gamma,\beta}(1,1) - \tilde{J}^{\gamma,\beta}(1,0)\right),$$

$$B_0 = 0,$$

$$C_i = \lambda\beta\left(\tilde{J}^{\gamma,\beta}(i+1,1) - \tilde{J}^{\gamma,\beta}(i+1,0) + \tilde{J}^{\gamma,\beta}(i,0) - \tilde{J}^{\gamma,\beta}(i,1)\right), \quad i \geq 1$$

$$C_0 = \lambda\beta\left(\tilde{J}^{\gamma,\beta}(1,1) - \tilde{J}^{\gamma,\beta}(1,0)\right),$$

$$D_i = \lambda\beta\left(\tilde{J}^{\gamma,\beta}(i+1,1) - \tilde{J}^{\gamma,\beta}(i,1)\right) - 1, \quad i \geq 1$$

$$E_i = \beta\mu_2\left(\tilde{J}^{\gamma,\beta}(i-1,1) - \tilde{J}^{\gamma,\beta}(i-1,0)\right), \quad i \geq 2$$

$$E_1 = 0.$$

Furthermore, the following observations are useful:

(i) A_i and D_i are increasing functions of i, $i \geq 0$, by the convexity of $\tilde{J}^{\gamma,\beta}(\cdot, x^2)$ (cf. Proposition 5 and (4.7)).

(ii) For every $i \geq 1, A_i + C_i = D_i$. This follows directly from the definition of $A_i,\ C_i, and\ D_i$.

(iii) For $i \geq 1, A_i \leq D_{i+1}$. This follows from Proposition 6 and (4.7), since

$$\tilde{J}^{\gamma,\beta}(i+1,1) - \tilde{J}^{\gamma,\beta}(i,0) \geq \tilde{J}^{\gamma,\beta}(i,1) - \tilde{J}^{\gamma,\beta}(i-1,0),$$

or

$$\tilde{J}^{\gamma,\beta}(i+2,1) - \tilde{J}^{\gamma,\beta}(i+1,1) \geq \tilde{J}^{\gamma,\beta}(i+1,0) - \tilde{J}^{\gamma,\beta}(i,0),$$

whence the assertion results.

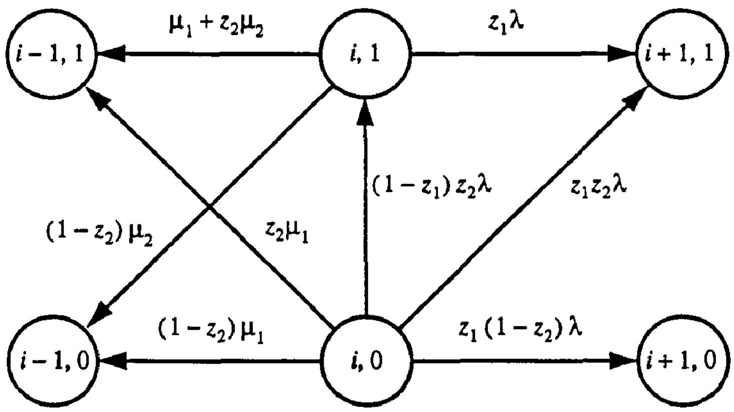

Figure 2: The state transition diagram

Referring to the state transition diagram in Figure 2, the dynamic programming equations can now be written as shown below. We remark that the slow channel, once activated, cannot be preempted at transition epochs corresponding to arrivals into the system, and departures from the fast channel. We omit the preliminary elementary algebra needed to arrive at (4.8) below.

For $i > 1$:

$$\begin{aligned} \tilde{J}^{\gamma,\beta}(i,0) =& 1+\gamma i + \min_{z^1,z^2\in\{0,1\}}\{z^1A_i + z^2B_i + z^1z^2C_i\}+ \\ & + \beta\left(\mu_1\tilde{J}^{\gamma,\beta}(i-1,0) + \lambda\tilde{J}^{\gamma,\beta}(i,0) + \mu_2\tilde{J}^{\gamma,\beta}(i,0)\right) \\ \tilde{J}^{\gamma,\beta}(i,1) =& 1+\gamma i + \min_{z^1,z^2\in\{0,1\}}\{z^1D_i + z^2E_i\}+ \\ & + \beta\left(\mu_2\tilde{J}^{\gamma,\beta}(i-1,0) + \mu_1\tilde{J}^{\gamma,\beta}(i-1,1) + \lambda\tilde{J}^{\gamma,\beta}(i,1)\right). \end{aligned} \tag{4.8a}$$

For $i = 1$, we get:

$$\begin{aligned} \tilde{J}^{\gamma\beta}(1,0) =& 1+\gamma + \min_{z^1,z^2\in\{0,1\}}\{z^1A_1 + z^2B_1 + z^1z^2C_1\}+ \\ & + \beta\left(\mu_1\tilde{J}^{\gamma,\beta}(0,0) + \lambda\tilde{J}^{\gamma,\beta}(1,0) + \mu_2\tilde{J}^{\gamma,\beta}(1,0)\right); \\ \tilde{J}^{\gamma,\beta}(1,1) =& 1+\gamma + \min_{z^1\in\{0,1\}}\{z^1D_1\} \end{aligned} \tag{4.8b}$$

$$+ \beta \left(\mu_2 \tilde{J}^{\gamma,\beta}(0,0) + \lambda \tilde{J}^{\gamma,\beta}(1,1) + \mu_1 \tilde{J}^{\gamma,\beta}(1,1) \right).$$

Finally, for $i = 0$, we have:

$$\tilde{J}^{\gamma,\beta}(0,0) = 1 + \min_{z^1, z^2 \in \{0,1\}} \{z^1 A_0 + z^1 z^2 C_0\} + \beta \tilde{J}^{\gamma,\beta}(0,0). \tag{4.8c}$$

The optimal control actions taken at states $(i,0)$ and $(i,1)$, respectively, are seen to appear in a nonlinear manner in the dynamic programming equations (4.8a)-(4.8c), and can be determined by the minimization with respect to (z^1, z^2) of the functions:

$$f_i^0(z^1, z^2) \triangleq z^1 A_i + z^2 B_i + z^1 z^2 C_i, \quad i \geq 0, \tag{4.9a}$$

and

$$f_i^1(z^1, z^2) \triangleq z^1 D_i + z^2 E_i, \quad i \geq 1. \tag{4.9b}$$

Before proceeding with the minimization, it is instructive to consider the nature of the optimal cost functions $\tilde{J}^{\gamma,\beta}(\cdot,0)$ and $\tilde{J}^{\gamma,\beta}(\cdot,1)$. From propositions 1-6 it is evident that the forms and relative values of these two cost functions will be as depicted in Figure 3a.

We commence with the actions taken at state $(i,0)$. There are four cases to be considered.

Case (i): $i < \bar{i} - 1$ (see Figure 3a). The values of $f_i^0(z^1, z^2)$ for the four possible choices of (z^1, z^2) are:

$$0 \text{ for } z^1 = 0, z^2 = 0;$$

$$A_i \text{ for } z^1 = 1, z^2 = 0;$$

$$B_i \text{ for } z^1 = 0, z^2 = 1;$$

$$A_i + B_i + C_i \text{ for } z^1 = 1, z^2 = 1.$$

Since in this case $B_i \geq 0$ and $B_i + C_i \geq 0$, the choice is between $(z^1 = 0, z^2 = 0)$ and $(z^1 = 1, z^2 = 0)$ according to whether $A_i \geq 0$ or $A_i < 0$. Thus, the optimal strategy disables the slower channel and accepts (resp. blocks) an incoming message at the buffer if $A_i < 0$ (resp. $A_i \geq 0$).

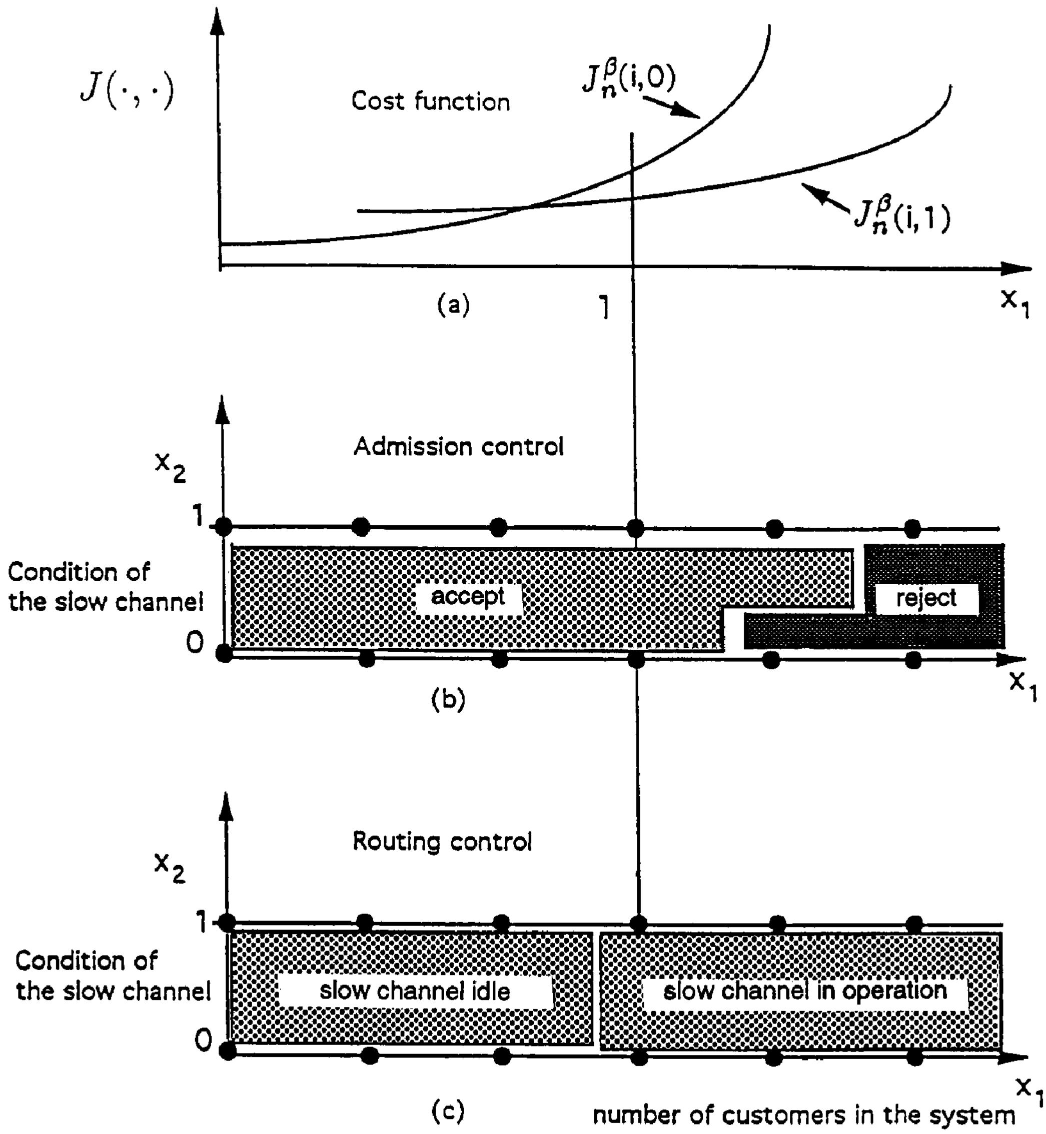

Figure 3: Structure of the optimal policy

Case (ii): $i > \bar{i}$: In this case it is easily verified that $B_i \leq 0$ and $B_i + C_i \leq 0$, so that the choice is between $(z^1 = 0, z^2 = 1)$ and $(z^1 = 1, z^2 = 1)$, the corresponding values of $f_i^0(z^1, z^2)$ being B_i and $A_i + B_i + C_i$, respectively. Thus, the optimal policy keeps the slower channel active and accepts (resp. blocks) an incoming message if $A_i + C_i < 0$ (resp. $A_i + C_i \geq 0$).

Case (iii) (resp. case(iv)): $i = \bar{i} - 1$(resp. $i = \bar{i}$). Based on the propositions of section 3, the admission control depends on the sign of A_i (resp. $A_i + C_i$) exactly as in the previous case.

The optimal control actions taken at states $(i, 1)$ are easily determined in a similar manner by the signs of D_i and E_i. If $D_i < 0$ (resp. ≥ 0), $E_i < 0$ (resp. ≥ 0), then $(z^1 = 1, z^2 = 1)$ (resp. $(z^1 = 0, z^2 = 0)$) minimize $f_i^0(z^1, z^2)$.

We are now in a position to characterize the optimal admission policy at the buffer; this is done in Propositions 7-10.

Proposition 7: The optimal admission policy is characterized by a switching curve (see Figure 3b).

Proof: We first show $z^1(i,0) = 1 \Rightarrow z^1(i',0) = 1$ for all $i' < i$, i.e., if the optimal policy admits an incoming message into the buffer at state $(i, 0)$, it must also do so at states $(i', 0)$, $i' < i$. Suppose $i \geq \bar{i}$ (cf. case (ii) above). If $z^1(i, 0) = 1$, then $A_i + C_i < 0$. From the convexity of $\tilde{J}^{\gamma,\beta}(\cdot, 0)$, it follows that for all $\bar{i} \leq i' \leq i$, $A_{i'} + C_{i'} < 0$, so that $z^1(i', 0) = 1$. Finally for $i' < \bar{i}$, again $A_{i'} < 0$ (by observation (*iii*) earlier in this section), whence $z^1(i,0) = 1$ (cf. case (i)). Similar arguments show that $z^1(i, 0) = 1$ for $i \leq \bar{i}$ would imply $z^1(i', 0) = 1$ for all $i' < i$.

Lastly, it follows in a straightforward manner that $z^1(i, 1) = 1$ implies $z^1(i', 1) = 1$ for $i' < i$. Indeed, if $z^1(i, 1) = 1$, then $D_i < 0$ whence $D_{i'} < 0$ for $i' < i$ by the convexity of $\tilde{J}^{\gamma,\beta}(\cdot, 1)$; consequently, $z^1(i', 1) = 1$.

Proposition 8: If the optimal policy accepts an incoming message at state $(i, 1)$, then it also does so at states $(i', 0), i' < i$, i.e., $z^1(i, 1) = 1$ implies $z^1(i', 0) = 1$.

Proof: Since $z^1(i, 1) = 1$, it follows that $D_{i'} < 0$ and $A_{i'} < 0$ for all $i' < i$, so that $z^1(i', 0) = 1$.

Proposition 9: For $i > \bar{i}$, $z^1(i, 1) = 0$ iff $z^1(i, 0) = 0$.

Proof: The proof is obvious by the fact that $D_i = A_i + C_i < 0$.

Finally, proposition 2 and corollaries 2,3 of section 3 lead to the following optimal routing strategy.

Proposition 10: The optimal routing policy is of the threshold type, i.e., there exists an integer $\bar{i}$ such that $z^2(i,\cdot) = 1(0)$ for $i \geq \bar{i}(i < \bar{i})$.

5. Convexity of the Discounted Optimal Cost

In this section we establish Propositions 5 and 6 (cf. section 3). We begin by considering Proposition 5, namely the convexity property of $J_*^{\gamma,\delta}(\cdot, x^2)$. In view of (4.3) and (4.7), it suffices to show for each $n \geq 0$, $0 < \beta < 1$, and $x^2 = 0, 1$, that

$$\tilde{J}_n^{\gamma,\beta}(x^1+1,x^2) - \tilde{J}_n^{\gamma,\beta}(x^1,x^2) \geq \tilde{J}_n^{\gamma,\beta}(x^1,x^2) - \tilde{J}_n^{\gamma,\beta}(x^1-1,x^2)$$

for all $x^1 \geq 1$. In order to do so, we shall employ a modification of the technique introduced in [4], wherein the discounted-cost, discrete time problem is first suitably transformed into a linear program.

Our approach entails artificially "enlarging" the state space of the system by redefining the state at instant k (corresponding to the transition epoch t_k) in terms of a triple $\mathbf{x}_k = (x_k^1, x_k^2, x_k^3)$, where

$x_k^1 =$ number of packets in the buffer and on channel 1 at the transition epoch t_k (and not the total number in the system, as defined earlier);

$x_k^2 =$ number of packets transfered from the buffer to channel 2 upto transition epoch t_k;

$x_k^3 =$ number of packets that have departed on channel 2 upto transition epoch t_k.

This new state description is intended solely for the proofs at this section, and should cause no confusion. It subsumes the state description of section 4, since the total number of messages in the systen at time instant k is given by $x_k^1 + x_k^2 - x_k^3$ while the condition of the second channel is simply $x_k^2 - x_k^3$. Clearly, $x_k^1 \geq 0$ and $x_k^2 - x_k^3$ belongs to $\{0, 1\}$. In terms of the new state description, the n-step β-discounted cost for the discrete-time problem with initial state x corresponding to a CS z in $\mathcal{P}$ is given by

$$\tilde{V}_n^{\beta}(\mathbf{x}, z) = E_{\mathbf{x}}^{z} \sum_{k=0}^{n-1} \beta^k \left((1 - z_k^1(\omega^k)) + \gamma(x_k^1(\omega^k) + x_k^2(\omega^k) - x_k^3(\omega^k)) \right), \qquad (5.1)$$

and the corresponding optimal cost by

$$\tilde{J}_n^{\gamma,\beta}(\mathbf{x}) = \min_{z\in\mathcal{P}} \tilde{V}_n^{\gamma,\beta}(\mathbf{x}, z). \tag{5.2}$$

Let A, D_1, D_2 represent, respectively, the events of an arrival of a packet at the buffer, and the departures of a packet on channels 1 and 2. Next, let $\Omega^k = \{\omega^k(\omega_1, \ldots, \omega_k) : w_i \in \{A, D_1 D_2\}\}$, $k = 1, 2, \cdots, n$ represent the collection of all events corresponding to arrivals and departures of packets at the transition epochs during the interval $[0, t_k]$. On Ω_k we define the following transition matrix:

$$\Xi_k(\omega^k) = \begin{cases} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \text{if } \omega_k = A \\ \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \text{if } \omega_k = D_1 \\ \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \text{if } \omega_k = D_2. \end{cases}$$

We can then write the evolution of the system state as

$$\begin{bmatrix} x_{k+1}^1(\omega^{k+1}) \\ x_{k+1}^2(\omega^{k+1}) \\ x_{k+1}^3(\omega^{k+1}) \end{bmatrix} = \begin{bmatrix} x_k^1(\omega^k) \\ x_k^2(\omega^k) \\ x_k^3(\omega^k) \end{bmatrix} + \Xi_{k+1}(\omega^{k+1}) \begin{bmatrix} z_{k+1}^1(\omega^{k+1}) \\ z_{k+1}^2(\omega^{k+1}) \\ z_{k+1}^3(\omega^{k+1}) \end{bmatrix}, \tag{5.3}$$

for $k = 0, 1, \cdots$, where $z_k^1(\omega^k)$ and $z_k^2(\omega^k)$ correspond, respectively (as in section 1), to the actions of admission to the buffer, and the activation of channel 2, at transition epoch t_k, and $z_k^3(\omega^k)$ takes the value 1 or 0 at t_k according to whether or not a "dummy" departure [6,12,17] occurs on channel 2. Equation (5.3) can be solved recursively to yield

$$\mathbf{x}_\mathbf{k}(\omega^k) = \mathbf{x} + \sum_{j=1}^{k} \Xi_j(\omega^j)\mathbf{z}_j(\omega^j), \tag{5.4}$$

where the notation $\mathbf{z}_k(\omega^k)$, $k = 1, 2, \cdots$, is obvious. We can now rewrite (5.1) as

$$\tilde{V}_n^{\beta,\gamma}(\mathbf{x}, z) = \mathbb{E}_\mathbf{x}^z \sum_{k=0}^{n-1} \beta^k \left(I(\omega^k = A)(1 - z_k^1) + \gamma(x_k^1 + x_k^2 - x_k^3) \right).$$

Upon substituting (5.4) in the equation above, we get

$$\tilde{V}_n^{\beta,\gamma}(\mathbf{x}, z) = \sum_{k=1}^{n} \sum_{\omega^k \in \Omega^k} c_k^1(\omega^k) z_k^1(\omega^k) + c_k^2(\omega^k) z_k^3(\omega^k) + c$$

where

$$c_k^1(\omega^k) = \mathbb{P}(\omega^k)\beta^k[\Xi_k^{11}(\omega^k)\gamma \sum_{j=k}^{n} \beta^j - I(\omega^k = A)],$$

$$c_k^2(\omega^k) = -\mathbb{P}(\omega^k)\beta^k \Xi_k^{33}(\omega^k)\gamma \sum_{j=k}^{n} \beta^j,$$

and

$$c = \lambda \sum_{k=1}^{n} \beta^k + \sum_{k=1}^{n} \gamma \beta^k [x^1 + x^2 - x^3].$$

The minimization problem (5.2) can now be transformed into the following linear program:

$$W_n(\mathbf{x}) = \min_{\{\mathbf{z}_k(\omega^k)\}_{k=1}^n} \sum_{k=1}^{n} \sum_{\omega^k \in \omega^k} c_k^1(\omega^k) z_k^1(\omega^k) + c_k^2(\omega^k) z_k^3(\omega^k) + c,$$

such that for each ω^k in Ω, and $k = 1, 2, \cdots, n$

$$z_k^i(\omega^k) \in [0,1], \qquad i = 1,2,3 \text{ (feasibility constraint)} \qquad \text{(LP)}$$

$$x^1 + \sum_{j=1}^{k} \Xi_j^{(1)}(\omega^j)\mathbf{z}_j(\omega^j) \geq 0,$$

and

$$0 \leq x^2 - x^3 + \sum_{j=1}^{k} (\Xi_j^{(2)}(\omega^j) - \Xi_j^{(3)}(\omega^j))\mathbf{z}_j(\omega^j) \leq 1$$

where $\Xi_j^{(i)}(\omega^j)$ denotes the ith row of the matrix $\Xi_j(\omega^j)$.

Remark: The first constraint is associated with the feasibility of the control actions. The second and third constraints, respectively, imply that the number of packets in the buffer and on the slow channel should be nonnegative, and, moreover, that the second channel cannot forward more than one packet at any time.

Lemma 1: For $0 < \beta < 1$, $n = 0, 1, \cdots$, $W_n^\beta(\cdot)$ is a convex piecewise linear function.

Proof: Since the quantity $\mathbf{x}$ enters linearly in the constraints of (LP), the lemma follows directly from the theory of linear programming [26, page 56].

Lemma 2: The linear program (LP) accepts an integer solution, i.e., $z_k^i(\omega^k)$ belongs to $\{0,1\}$, $i = 1,2,3$ for every $k = 1, 2, \cdots, n$.

Proof: We denote by z^* the optimal strategy that solves the linear program (LP). By using duality theory [26, page 50] as it applies to linear programming, we conclude that z^* is an optimal solution for (LP) iff there exist suitable vector-valued variables $\underline{\lambda}_k^*(\omega^k)$ in $\mathbb{R}^3$ such that $\underline{\lambda}_k^*(\omega^k) \geq 0$ (componentwise), and the following conditions are satisfied for $k = 1, 2, \ldots, n$, ω^k in Ω^k: (We drop below the dependence of certain variables on ω^k to make the presentation simpler.)

c1) z^* is a solution to the following program:

$$\min_z \sum_{k=1}^{n} \sum_{\omega^k} \left(c_k^1 z_k^1 + c_k^2 z_k^3 - \lambda_k^{*1} x_k^1 - \lambda_k^{*2}(x_k^2 - x_k^3) + \lambda_k^{*3})(x_k^2 + x_k^3 - 1) \right).$$

c2) The state trajectory generated by z^*, denoted $\mathbf{x}_k(z^*)$, should satisfy,

$$x_k^1(z^*) \geq 0 \quad \text{and} \quad 0 \leq x_k^2(z^*) - x_k^3(z^*) \leq 1.$$

c3) If $\lambda_k^{*1} > 0$, then $x_k^1 = 0$. Further, if $\lambda_k^{*2(3)} > 0$ then $x_k^2 + x_k^3 = 0(1)$.

The cost function in c1) can be transformed (after a simple change of the variables of summation) to:

$$\min_z \sum_{k} \sum_{\omega^k} \left(c_k^1 z_k^1 + c_k^2 z_k^3 - \sum_{j=k}^{n} \lambda_j^{*1} \right) \Xi_k^{(1)} \mathbf{z}_k$$

$$+ (\sum_{j=k}^{n} (\lambda_j^{*3} - \lambda_j^{*2})(\Xi_k^{(2)} - \Xi_k^{(3)}) \mathbf{z}_k + \text{terms independent of } z$$

$$= \min_z \sum_{k=1}^{n} \sum_{\omega^k} \mathbf{d}_k(\mathbf{c}_k, \underline{\lambda}_k^*, \omega_k) \mathbf{z}_k + \text{terms independent of } z$$

where $\mathbf{d}_k(\mathbf{c}_k, \underline{\lambda}_k^*, \omega_k)$ is defined in an obvious manner.

We then conclude immediately that

$$\mathbf{z}_k^{*i} = \begin{cases} 1 & \text{if } d_k^i(\mathbf{c}_k, \underline{\lambda}_k^*, \omega_k) < 0 \\ 0 & \text{if } d_k^i(\mathbf{c}_k, \underline{\lambda}_k^*, \omega_k) > 0 \\ \in [0,1] & \text{if } d_k^i(\mathbf{c}_k, \underline{\lambda}_k^*, \omega_k) = 0 \end{cases} \quad i = 1, 2, 3\,. \tag{5.5}$$

Henceforth, suppose that the initial condition $\mathbf{x}$ is integer-valued. For $k = 1, 2, \cdots, n$ let $z^*, \underline{\lambda}^*$ satisfy the optimality conditions c1), c2) and c3). We shall use z^* to construct an integer-valued policy z that is optimal, i.e., satisfies the abovementioned conditions. To this end, we provide the following lemma.

Lemma 3: Consider the following region in $\mathbb{R}^3$:

$$X = \left\{ p_1 e_1 + p_2 e_2 + p_3 e_3,\ p_i \in \left(-\frac{1}{2}, \frac{1}{2} \right] \right\},$$

where $e_1 = (1,0,0)^T, e_2 = (-1,+1,0)^T, e_3 = (0,0,-1)^T$. Let $\xi^i(\omega), i = 1,2,3$ be the ith column of the matrix $\Xi(\omega)$. Then

$$\{X + \Xi(\omega)\mathbf{z},\ \ \mathbf{z} \in [0,1]^3\} \subset X \cup \left\{ \bigcup_{\substack{i,j \in \{1,2,3\}, i \neq j \\ \omega \in \{A, D_1, D_2\}}} X + \xi^i(\omega) + \xi^j(\omega) \right\}.$$

Proof: The proof is straightforward and, hence, omitted.

Proposition 11: There is an integer-valued (i.e., $\{0,1\}$-valued) policy $z = \left(z_k(\omega^k), k = 1, 2, \ldots n\right)$ such that $\mathbf{z}_k^*(\omega^k) = \mathbf{z}_k(\omega^k)$, where the latter is integer-valued, and for all ω^k in Ω^k and $k > 1$, it holds that

$$\Delta_k \triangleq \left(\mathbf{x}_k(\omega^k, z^*) - \mathbf{x}_k(\omega^k, z)\right) \in X.$$

Proof: The proof is by induction. Suppose for some $k \geq 0$ that Δ_k is in X. Then, it follows that

$$\Delta_{k+1} = \Delta_k + \Xi_{k+1}(\omega^{k+1})\mathbf{z}_{k+1}^*(\omega^{k+1}) - \Xi_{k+1}(\omega^{k+1})\mathbf{z}_{k+1}(\omega^{k+1}).$$

If $z^*_{k+1}(\omega^{k+1})$ is integer-valued, then obviously $z^*_{k+1} = z_{k+1}$. Else, by lemma 1, either $(\Delta_k + \Xi_{k+1}(\omega^{k+1})\mathbf{z}_{k+1}(\omega^{k+1}))$ is in X, so that we set $z^i_{k+1} = 0 \ (i = 1,2,3)$, or $(\Delta_k + \Xi_{k+1}(\omega^{k+1})\mathbf{z}_{k+1}(\omega^{k+1}))$ is in $\Big(X + \xi^i_{k+1}(\omega^{k+1}) + \xi^j_{k+1}(w^{k+1})\Big)$, when we choose $z^i_{k+1} = z^j_{k+1} = 1$ and $z^\ell_{k+1} = 0, \quad \ell \neq i,j$. In either case Δ_{k+1} belongs to X.

Proposition 12: The integer-valued policy z constructed above is optimal, i.e, it satisfies conditions c1), c2) and c3).

Proof: Condition c1) is trivially satisfied, since $z^i_k(w^k) = z^{*i}_k(w^k)$ whenever z^{*i}_k is integer valued. We now check the feasibility conditions c2). We wish to show that $x^1_k \geq 0$ and $x^2_k - x^3_k \geq 0$. Suppose this were not true. Then, since z^1_k, z^2_k, z^3_k equal 0 or 1, we clearly have $x^1_k \leq -1$ and $x^2_k + x^3_k \leq -1$. Since Δ_k lies in X, we conclude that $x^{*1}_k = x^1_k + p_1 - p_2$ for p_1, p_2 in $(\,-\frac{1}{2}, \frac{1}{2}\,]$. Hence, $x^{*1}_k < -1 + \frac{1}{2} + \frac{1}{2} = 0$, which is clearly a contradiction since x^{*1}_k is known to be optimal (and hence feasible).

Similarly, since Δ_k belongs to X, we readily see that:

$$\left(x^{*2}_k, x^{*3}_k\right) = \left(x^2_k, x^3_k\right) + (p_2, p_3) \quad p_1, p_2 \epsilon (\,-\frac{1}{2}, \frac{1}{2}\,]$$

and

$$x^{*2}_k - x^{*3}_k \leq -1 + p_2 - p_3 < -1 + \frac{1}{2} + \frac{1}{2} = 0,$$

which again lead to a contradiction.

Next, we must show that $x^2_k - k^3_k \leq 1$. As before, if this were not true we must have $x^2_k - x^3_k \geq 2$. Using the same arguments as before, $x^{*2}_k - x^{*3}_k \geq 2 + p_2 - p_3 > 2 - \frac{1}{2} - \frac{1}{2} > 1$, clearly a contradiction.

Finally, we establish the complementary slackness conditions c3). It is enough to show that if $x^{*1}_k = 0$ then $x^1_k = 0$. In a similar way we must show that $x^{*2}_k - x^{*3}_k = 0(1)$ implies $x^2_k + x^3_k = 0(1)$. As before, we have that $x^1_k = x^{*1}_k - (p_1 - p_2) = p_2 - p_1$ belongs to $(-1, 1)$ so that $x^1_k = 0$, since x^1_k is integer-valued. All other cases in c3) can be treated in a similar manner.

At this point, the proof of proposition 5 is evident. Returning to the old state description of the system, if $\mathbf{x} = (x^1, x^2)$ is a point with integer coordinates, then since z, the solution of the linear program (LP), is integer-valued, i.e., belongs to $\{0,1\}$, and $J_n^{\beta,\gamma}(x^1,x^2)$ is unique for each n, we conclude that:

$$\tilde{J}_n^{\beta,\gamma}(x^1,x^2) = W_n(x^1 - x^2, x^2, 0),$$

for $x^1 \geq 1$, and x^2 in $\{0,1\}$; furthermore, $\tilde{J}_n^{\beta,\gamma}(x^1,x^2)$ inherits the convexity (with respect to the argument x^1) of $W_n(x^1 - x^2, x^2, 0)$ for every n. Hence, $\tilde{J}^{\beta,\gamma}(x^1,x^2)$ is also convex with respect to x^1.

Next, from [26, page 56] we have that $W_n^{\beta,\gamma}(x^1,x^2,0)$ is a piecewise linear function. Furthermore, by using arguments similar to those in Proposition 1, it can be shown that it is an increasing function in x^1 and x^2, and hence it holds that

$$W_n(x^1+1,1,0) - W_n(x^1+1,0,0) \geq W_n(x^1,1,0) - W_n(x^1,0,0). \qquad (5.6)$$

Proposition 6 now follows immediately. Property (5.6) is known as "supermodularity"; a detailed proof of (5.6) can be found elsewhere [8,27].

6. The Average Cost Problem

The average cost problem (P1) entails a minimization of the expected average cost per unit time [4,15,22,25]. Our elaborate approach below is necessitated by the fact that the state space is not finite. We determine the optimal stationary policy for the average cost problem (P1) by associating it with the discounted cost problem (P2). Moreover, the optimal average cost for problem (P1) can be expressed as follows by using the standard procedure of uniformization earlier employed in section 4; thus

$$\tilde{J}_{av}(\mathbf{x}) = \min_{z \in \mathcal{P}} \tilde{V}(\mathbf{x}, z)$$

and

$$\tilde{V}(\mathbf{x}, z) = \limsup_{n \to \infty} \frac{1}{n} \mathbb{E}_{\mathbf{x}}^{z} \left(\sum_{k=0}^{n} 1 - z_k^1(\mathbf{x}_k) + \gamma x_k^1 \right)$$

where, with a slight abuse of notation, z denotes a policy that is not necessarily stationary.

However, the following lemma proposes a stationary policy which is a candidate for average-cost optimality. Furthermore, this stationary policy will be seen to arise as a limit of optimal policies associated with the discounted cost problem (P2). The lemma can be found in [4] and [22].

Lemma 4: Let $\{\beta_n\}_{n=1}^{\infty}$ be a sequence of discount factors converging to 1. Let $\{z_{\beta_n}\}_{n=1}^{\infty}$ be the associated sequence of (stationary) optimal policies for the discounted-cost problem. Then there exists a subsequence $\{\beta_{n'}\}$ and a stationary policy z which is the limit point of $z_{\beta_{n'}}$.

Althought the lemma has been proved in [4,22], for the sake of completeness we present a brief proof below along with some observations.

Proof of lemma 4: The finiteness of the action set $\mathcal{D} = \{0,1\}^2$ enables it to be viewed as a compact topological space with a discrete topology, where every subset of $\mathcal{D}$ is simultaneously open and closed. Further, the associated topological basis formed by these open sets is finite. By the Tychonoff theorem [21], the countable product space $\mathcal{A} = \mathcal{D}^{\mathcal{S}}$ is also compact under the product discrete topology. Since the basis for $\mathcal{A}$ under the same topology is countable and since $\mathcal{A}$ is normal [21], it is also metrizable by Urysohn's lemma [7]. Consequently, $\mathcal{A}$ is sequentially compact, i.e., every sequence $\{z_{\beta_n}\}$ (of stationary policies) has a convergent subsequence $\{z_{\beta_{n'}}\}$ converging to a stationary policy z in the following sense: for every $\mathbf{x}$ in $\mathcal{S}$ there exists an integer $N(\mathbf{x})$ such that $z_{\beta_{n'}}(\mathbf{x}) = z(\mathbf{x})$ for $n' \geq N(\mathbf{x})$.

Next, we establish that the stationary policy z of the previous lemma will have the form derived in section 4. In particular, we show that $z^1(x^1, x^2) = 0$ implies $z^1(\bar{x}^1, x^2) = 0$ for $\bar{x}^1 \geq x^1$. If this were not true, suppose that $z^1(\bar{x}^1, x^2) = 1$ for $\bar{x}^1 > x^1$. Then since $z_{\beta_{n'}} \to z$, we conclude that there exist $N(x^1, x^2)$, $\bar{N}(\bar{x}^1, x^2)$ such that $z^1_{\beta_{n'}}(x^1, x^2) = 0$ for all $n' \geq N$, and $z^1_{\beta_{n'}}(\bar{x}^1, x^2) = 1$ for all $n' \geq \bar{N}$. By choosing $k = max\{N, \bar{N}\}$ we see that $z^1_{\beta_k}(x^1, x^2) = 0$ and $z^1_{\beta_k}(\bar{x}^1, x^2) = 1$ for $\bar{x}^1 \geq x^1$ which contradicts proposition 7 of section 4. A similar argument can be applied to the routing control z^2.

It only remains to establish the optimality of the stationary policy z of lemma 4 for the average cost problem. To this end, we consider the following two cases determined by the nature of the admission control z^1.

1) Assuming that $z^1(x^1, x^2) = 0$ for some finite x^1, we conclude that the underlying Markov decision process is "essentially" a finite, irreducible chain and, hence, ergodic. If $p_z(\mathbf{x})$ is the associated stationary probability distribution under the policy z we obviously have $\sum_{\mathbf{x}\in S} p_z(\mathbf{x})(1 - z^1(\mathbf{x}) + \gamma x^1) < \infty$. Moreover, since $\tilde{J}^{\gamma,\beta}(x^1, x^2)$ is increasing in x^1, x^2 (Propositions 1,3), we have that $\tilde{J}^{\gamma,\beta}(x^1, x^2) - \tilde{J}^{\gamma,\beta}(0,0) \geq 0$ for all β, γ, x^1, x^2. Hence the following theorem from [22] follows:

Theorem 2: The policy z from Lemma 4 is optimal for the average cost problem (P1). Furthermore the average cost is given by:

$$\tilde{J}_{av} = \lim_{\beta\to 1}(1-\beta)\tilde{J}^{\beta,\gamma}(\mathbf{x}),$$

not depending on the initial state $\mathbf{x}$.

2) Assuming that $z^1(x^1, x^2) = 1$ for all x^1, the problem reduces to that studied by Lin and Kumar [10], where the routing control is shown to be of the threshold type.

7. Concluding Remarks

Considering the admission control problem alone, Stidham [24] has established, under Poisson arrivals and exponential service times, the optimality of the threshold policy in terms of minimizing a combination of blocking and delay costs. For the same arrival and service distributions, Lin-Kumar [10] and Walrand [29] show that a routing policy of the threshold type minimizes a delay cost. A natural question that arises, therefore, concerns the possibility of separating the joint problem of identifying admission and routing controls which minimize a weighted blocking and delay cost, into the two separate problems addressed above. The dynamic programming equations (4.8) of section 4 involve a strong correlation between the two controls, and suggest no *a priori* separation. It is interesting, however, that the final *forms* of the joint controls turn out to be of the threshold type, which forms are individually optimal for the two separate problems. Of particular note is the optimality of the threshold routing policy even after the exogenous Poisson arrival process has been "filtered" by the admission control. The approach in [10], as also ours, requires the exogenous arrival process to be Poisson and the transmission times to be exponential. Otherwise the dynamic programming equations cannot be written in their present form. The approach in [29] is also based on an exponential assumption.

Finally we remark that the problem which seeks a minimal blocking probability under an explicit constraint on the average delay in the system is as yet unsolved. On the basis of Ross [18], it may be conjectured that the optimal policy will be randomized rather than stationary.

Appendix 1

We introduce two different systems both with the same total number of customers; at $t = 0$, the faster channel is idle in the first system, while being active in the second. An optimal policy corresponding to the first sytem is then applied to both systems. Using the same technique as in Walrand [29, Lemma 3.1 (1,2), p. 132], it can be deduced that the (second) system which activates the faster channel incurs a (strictly) lower cost than the other.

Appendix 2

Observe from [11, Theorem 1], that $J^{\gamma,\delta}(\cdot)$ is the unique solution to the following functional dynamic programming equation:

$$J^{\gamma,\delta}(\mathbf{x}) = \min_{z \in \mathcal{P}_S} \left(1 - z^1(\mathbf{x}) + \gamma x^1 + \sum_{\mathbf{x}' \in S} \beta_\delta(\mathbf{x}, z, \mathbf{x}') J^{\gamma,\delta}(\mathbf{x}') Pr(\mathbf{x}'|\mathbf{x}, z) \right) \quad (A2.1)$$

where $\mathbf{x} = (x^1, x^2)$, and $\beta_\delta(\mathbf{x}, z, \mathbf{x}')$ is an expected discount factor of the form

$$\beta_\delta(\mathbf{x}, z, \mathbf{x}') = \int_0^\infty e^{-\delta\xi} dT(\xi|\mathbf{x}, z, \mathbf{x}'),$$

with $T(\cdot|\mathbf{x}, z, \mathbf{x}')$ denoting the probability distribution function of the random time it takes the system to go from state $\mathbf{x}$ to $\mathbf{x}'$ under the CS z.

Next, using dynamic programming arguments, it can be easily shown that $J_n^{\gamma,\delta}(\cdot)$ satisfies the following recursion:

$$J_{n+1}^{\gamma,\delta}(\mathbf{x}) = \min_{z\in\mathcal{P}} \left(1 - z^1(\mathbf{x}) + \gamma x^1 + \sum_{\mathbf{x}'\in S} \beta_\delta(\mathbf{x}, z, \mathbf{x}') J_n^{\gamma,\delta}(\mathbf{x}') Pr(\mathbf{x}'|\mathbf{x}, z)\right). \tag{A2.2}$$

Since $J_{n+1}^{\gamma,\delta}(\cdot) \geq J_n^{\gamma,\delta}(\cdot)$, we have that $J_\infty^{\gamma,\delta}(\mathbf{x}) = \lim_{n\to\infty} J_n^{\gamma,\delta}(\mathbf{x})$ exists. Moreover $J_\infty^{\gamma,\delta}(\mathbf{x})$ is the unique solution to the contraction mapping (A2.2). This observation, together with the fact that $J^{\gamma,\delta}(\mathbf{x})$ is the unique solution to the same contraction mapping (A2.1), yields that $J_\infty^{\gamma,\delta}(\mathbf{x}) = J^{\gamma,\delta}(\mathbf{x})$.

Appendix 3

Proof of Proposition 2: Suppose that for some $\delta > 0$ and for all x the reverse inequality holds (with respect to (3.1)), i.e.,

$$J^{\gamma,\delta}(x, 0) < J^{\gamma,\delta}(x, 1).$$

We shall show that this supposition leads to a contradiction. The proof employs coupling arguments *a la* Walrand [29]. Consider a system with initial state $(x, 0)$, where x is a positive integer. Consider a second system which is similar but has initial state $(x, 1)$. Couple the arrival and service processes of the two systems, and apply to each the optimal strategy, denoted $z = (z^1, z^2)$, associated with the first system (i.e., with initial state $(x, 0)$). Let $(\mathbf{x}_t)$ and $(\tilde{\mathbf{x}}_t)$ respectively represent the corresponding state trajectories. By the supposition above, observe that in view of the stationarity of z, the first system never forwards a message through the slower channel.

Letting $V^{\gamma,\delta}(x,1) = \mathbb{E}_{(x,1)}^z \left(\int_0^\infty e^{-\delta t}(1 - z_t^1 + \gamma \tilde{x}_t^1)dt\right)$, it is clear that $J^{\gamma,\delta}(x,1) \leq V^{\gamma,\delta}(x,1)$. Let $\tau = \min(t : x_t^1 = 0)$ and let σ be an exponential random variable with mean μ_2^{-1} which represents the packet transmission time on the slower channel. Then

$$\begin{aligned} J^{\gamma,\delta}(x,0) - J^{\gamma,\delta}(x,1) &\geq J^{\gamma,\delta}(x,0) - V^{\gamma,\delta}(x,1) \\ &= \gamma\, \mathbb{E}_{(x,0)}^z (1[\sigma \leq \tau] \int_\sigma^\tau e^{-\delta t} dt - 1[\sigma > \tau] \int_0^\sigma e^{-\delta t} dt) \\ &\triangleq \varphi(x, \delta). \end{aligned}$$

Next consider an $M|M|1$ system starting with x initial packets, with *no arrivals*, and with the service time distribution being exponential with parameter μ_1. Let $\tilde{\tau}$ denote the time at which this system empties. Define

$$\tilde{\varphi}(x,\delta) = \gamma\,\mathbb{E}(1[\sigma \leq \tilde{\tau}] \int_{\sigma}^{\tilde{\tau}} e^{-\delta t} dt - 1[\sigma > \tilde{\tau}] \int_{0}^{\sigma} e^{-\delta t} dt).$$

Since $\tau \geq_{\text{st}} \tilde{\tau}$, it is clear that $\varphi(x,\delta) > \tilde{\varphi}(x,\delta)$ for all x. Then our supposition is contradicted if we show that $\tilde{\varphi}(x,\delta)$ is non-negative for some x suitably large. To this end, we observe that as x increases $\tilde{\tau}$ increases stochastically, so that $1[\sigma \leq \tilde{\tau}]$ increases while $1[\sigma > \tilde{\tau}]$ decreases, both in the stochastic sense. Noting that $\mathbb{P}(\sigma > \tilde{\tau}) = 1 - \mathbb{P}(\sigma \leq \tilde{\tau})$ goes to 0 as x increases, it is clear that there exists an integer $\overline{x} = \overline{x}(\delta)$ such that $\tilde{\varphi}(x,\delta) > 0$.

Proof of Corollary 2: We proceed in a manner similar to that of Proposition 2. For each integer $x_k \geq 1$, an integer $x_{k+1} > x_k$ can be determined in an inductive manner by redefining at each step τ =min$(t : x_t^1 = x_k)$ and proceeding as in Proposition 2.

Proof of Corolarry 3: The proof is as in [29, p.133, case 3]. Consider two systems with initial conditions $(x,0)$ and $(x,1)$. As long as the total number of messages lies between x_1 and x_2, *only* the faster channel is employed. It is then straightforward to show that the second system incurs a lower cost by using the slow channel at $t = 0$ instead of ing it at some later random time.

Proof of Proposition 4: The proof is similar to that of the previous proposition. Let z be the optimal policy associated with the system starting with initial condition $(x+1,1)$. If σ is an exponential random variable with mean μ_2^{-1}, we easily get:

$$J_n^{\gamma,\delta}(x+1,1) - J_n^{\gamma,\delta}(x,0) \geq \gamma\,\mathbb{E}^z_{(x+1,1)} \int_0^{\tau} e^{-\delta t} dt \geq 0$$

where $\tau = \min(t_n, \sigma)$. Similar arguments hold true for the case $n \to \infty$.

Proof of Proposition 5: We use the same argument as in [29, p. 133]. Let σ_1 and σ_2 be random variables representing the transmission times of a packet on the fast and slow channels respectively. Clearly, we can choose $\sigma_2 = \frac{\mu_2}{\mu_1}\sigma_1$. Consider two similar systems, the first starting with initial condition (1,1), and the second with initial condition (1,0). Denote by z the optimal policy associated with the first system. For the second system, we follow the policy $\tilde{z}$ constructed as follows: Whenever z activates the fast channel (i.e., accepts messages in the system), $\tilde{z}$ enables the slower channel.

We need only consider two cases. In the first case, the fast channel of the second system is transmitting, and hence, so is the slow channel of the first system. Then the strategies z and $\tilde{z}$, as defined above, will result in the same state trajectories for the two systems, and hence, the two systems will incur identical costs. In the second case, the fast channel of the second system is idle while the slow channels of both systems are busy. By introducing a dummy packet on the fast channel of the second system, the system states are coupled. As the dummy packet incurs no cost, the assertion is established in this case too.

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